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Quality Innovation: Driving Forces and Implications for Production, Trade, and Consumption

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Quality Innovation: Driving Forces and Implications for Production, Trade, and Consumption

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to my beloved wife and my dear parents

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Quality Innovation: Driving Forces and Implications for Production, Trade, and Consumption

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The dissertation has three main chapters on product quality innovation. First, we compare innovation effort and social welfare between monopoly, duopoly, and the social planner in a dynamic model with quality dependent on a continuous know-how stock. The technology frontier–the largest reachable know-how stocks–does not always positively depend on competitiveness, i.e. a duopoly may technologically surpass the social planner. However, social welfare is always positively tied to competitiveness. Second, with a general equilibrium model, we derive a relative price function expressing productivity and quality effects, and develop a method for inferring relative quality changes. An application to services versus goods of the US from 1946-2006 provides evidence on aggregate quality changes and suggests us to incorporate quality variations when explaining relative prices. Third, we build a two-product model where productivity changes lead to reallocations of labor between quantity production and quality innovation. The correlation between relative productivity and relative quality is negative for low-range substitutability and positive for medium-range substitutability between the two products. Looking at services versus goods of the US, the correlation is negative and productivity-driven quality can play a significant role in general quality development.

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Chapter 1

Introduction

As consumers, we get more utility from larger quantities and better quality of goods and services. In the current dissertation, if a is the quantity and α is the quality of some product, the *effective quantity* of consumption is $c = \alpha \times a$. In this specification, if either quantity or quality is zero, we consume nothing. For example, human beings cannot survive for an extended period of time just by drinking pure water. We will cover three models on quality innovation (changes in α) where the distinction between quantity and quality matters. Respectively, we need the distinction when: (i) firms compete to gain customers with better quality; (ii) we want to explain some relative price data; and (iii) the economy as a whole needs to find the best labor reallocation between quantity production and quality innovation when productivity changes. The rest of the dissertation is structured as follows.

Chapter 2 considers the effects of market structure on quality innovation effort and social welfare. We compare three allocation mechanisms in a model of dynamic quality innovation: monopoly, duopoly, and the social planner. The new feature in this model relative to the technological progress literature is to make quality advances depend upon a continuous know-how stock. In contrast to the previous models, this framework allows for more flexible innovation strategies and direct comparisons of technology frontiers which show the largest reachable know-how stocks. The chapter relies on analytical and computational approaches to compare technology frontiers and social welfare under the different allocation mechanisms. When products are perfectly substitutable, the technology frontier is highest under the social planner, lower under duopoly, and lowest under monopoly. However, when products are less substitutable, a duopoly may surpass the technology frontier under the social planner. *Ex ante* and long-run social welfare are always highest under the social planner and lowest under monopoly.

Chapter 3 develops a simple general equilibrium model to infer relative quality changes at the aggregate level and applies the inference method to the goods and services sectors of the United States from 1946 to 2006. The data are from the US National Income and Product Accounts. The existing empirical quality literature embraces the idea that price information alone can be used to track quality changes. However, prices are also driven by other forces, especially quantities supplied to markets. In solving this identification problem, the general equilibrium model generates an equilibrium relative price function which can be decomposed into productivity and quality effects. A method for inferring relative quality changes using aggregate data is then derived from this decomposition. In applying this method, the chapter finds first that the quality of US services relative to US goods was decreasing after 1946 and has been increasing since the 1970s. Second, in the services sector, productivity and quality, relative to the goods sector, are negatively correlated, suggesting an endogenous link between the two. Third, without quality variations, productivity changes alone cannot fully explain the evolution of the relative price of services. This suggests that ignoring quality variations when explaining relative prices can lead to incorrect conclusions. Being quite simple, the quality inference method can also be applied to many other countries.

Chapter 4 considers a fixed set of two products whose quality varies in response to changes in productivity (productivity-driven quality) in the context of competitive growth and business-cycle models. The growth model is then fitted to US data and generates a time path of relative quality changes which are hypothetically subject only to productivity variations. The chapter addresses two questions: First, how does productivity affect quality? Second, in the United States, how important is productivity-driven quality in total quality, where the latter varies due to all possible sources like randomly developed ideas for quality improvement as well as productivity? In this model, labor is used for both quantity production and quality innovation. As quantity and quality can be substitutable, a change in productivity will induce a reallocation of labor, leading to quality variations. Thus, differing from the variety-growth and quality-ladder literature, the chapter does not use market power to explain why quality varies. Theoretically, we find that the correlation between (relative) productivity and (relative) quality depends on two key parameters, which govern how substitutable the products are (substitution parameter) and how easy it is to improve quality (innovation parameter). Specifically, productivity and quality have a negative correlation for low-range substitutability and a positive correlation for medium-range substitutability, where the upper bound of the medium range negatively depends on the innovation parameter. The model is then applied to the US services-goods economy from 1970 to 2006 using aggregate data from the US National Income and Product Accounts. The parameter estimates imply a negative correlation between productivity and quality. In addition, productivity-driven quality can play a significant role in total quality.

Chapter 2

Competitiveness and Quality Innovation

2.1 Introduction

Intel and AMD are racing against each other to improve the speed of Pentium and Athlon micro processors, respectively. Even though Pentium and Athlon more or less do the same thing, they have their own patent protection. Quality innovation in the form of higher computational speeds is a clear example of technological progress. Between the two, the faster processor pleases consumers more, having supremacy over its competitor. We can find similar instances with a small number of dominant firms who repeatedly improve their product quality in the commercial jet market, the cellular phone market, etc.

Without races, technology may not progress as fast as observed. However, quality improvements often come at huge Research and Development (R&D) costs, and races may lead to wasteful investments. This raises the question: how does market structure affect quality innovation efforts and social welfare? We will consider three allocation mechanisms in a model of dynamic quality innovation: monopoly, duopoly, and the social planner. In this model, firms spend on R&D to have superior blueprints (know-how) according to which new and better product generations are produced. In this sense of technological progress, know-how stocks do not depreciate, and innovation steps can be continuous. Innovation efforts mean both how much firms spend on R&D and how much know-how they will accumulate. We are interested in both *ex ante* discounted life-time and maximal long-run social welfare. The terms *maximal* and *frontier* equivalently mean the know-how boundary beyond which firms no longer innovate.

We are addressing an important and interesting question. First, it is widely believed that technological progress is important for improving quality of life. Thus, understanding the innovation behaviors of different allocation mechanisms helps us design policies for better outcomes. Second, answers to different aspects of this question are not obvious at the face value. For example, if product quality never depreciates, does a monopoly have an incentive to innovate at all, as it does not face any competition? Do races for supremacy improve social welfare as the speed of quality innovation accelerates, whereas more resources are spent on R&D rather than on consumption? Without quality depreciation, do we always see a duopoly market dominated by one firm and the laggard never catch up with the front-runner? Third, as firms can choose continuous steps to progress, how does the innovation dynamics of different market structures, especially that of the duopoly, look like?

There are some highlights about the methodology. First, know-how stocks, or alternatively quality levels, constitute the state in the model. As mentioned above, this state is endogenously driven and non-decreasing. For tractability, we assume there is a threshold of know-how stock beyond which no firms can raise quality. Second, firms can choose continuous steps rather than discrete ones to push up quality, and they face uncertainty in realizing those steps. The introduction of continuous innovation steps allows for continuous price ratios and enriches R&D competition strategies in the duopoly. Third, behavior of the duopoly relies on some equilibrium concepts. Every period, firms engage in price competition or a Bertrand game. In addition, they base their R&D efforts on the state, and interact according to the pure-strategy first-order Markov Perfect Equilibrium (MPE). Fourth, the dynamic game setup does not support analytical solutions, and needs to rely on numerical characterizations. The threshold assumption helps us know the solution far into the future. Based on this knowledge, a backward-induction numerical algorithm is developed to solve the dynamic game. Fifth, to facilitate welfare analyses, consumer utility is quasi-linear, absorbing firms' profits.

Here are the main findings. First, when products are perfectly substitutable, the technology frontier is highest under the social planner, lower under duopoly, and lowest under monopoly. In addition, in a not-too-old duopoly, innovation investments are intensified when firms are neck-and-neck and alleviated when firms are far apart. Second, when products are less substitutable, a duopoly may follow an unbalanced evolution path and surpass the technology frontier under the social planner. Third, *ex ante* and long-run social welfare are highest under the social planner and lowest under monopoly.

The vast game-theoretic R&D literature related to this study can be divided into three overlapping groups of representative studies: (i) *patent races* with Scherer (1967), Loury (1979), Dasgupta & Stiglitz (1980), Lee & Wilde (1980), Harris & Vickers (1985, 1987), Reinganum (1981, 1982), Grossman & Shapiro (1987), Lippman & McCardle (1987), Klette & Griliches (2000), and Doraszelski (2003); (ii) *technology ladder* with Griliches (1979), Segerstrom et al. (1990), Aghion & Howitt (1992), Grossman & Helpman (1991), and Aghion et al. (2001); and (iii) *MPE industry dynamics* with Pakes & McGuire (1994, 2001), Ericson & Pakes (1995), and Doraszelski & Markovich (2007).

The current study deviates from those groups in several aspects. First, while the patent races set fixed prizes for R&D races, our model explicitly specifies profits as product market outcomes, and quality innovation is repeatedly driven by the desire for more and sustainable profits. Second, technology ladders are often in the form of quality ladders on which the front-runner and many laggards are one step apart. Effectively, competitors only choose the probability to progress one step. In our model, a laggard may choose a large step and surpass the front-runner in the next period if the R&D project succeeds. Aghion et al. (2001) allow competitors to be a number of steps apart. However, the laggard has to catch up with the front-runner before fighting for future leadership. In addition, they model a race down the production cost ladder which is naturally bounded from below by zero and has downward price effects. Our setup implies that quality innovation can bring about both larger market shares and higher prices. Third, the current MPE industry dynamics models allow net-state variables to move (exogenously) backward as well as (endogenously) forward, and eventually cycle in an ergodic set. Conceptually, no firm will dominate forever. In addition, without entry and exit, a laggard may catch up some day with only a little effort. Our model has a non-decreasing state and allows for the possibility of permanent dominance.

The rest of the chapter is structured as follows. Section 2.2 lays out the primitives of the environment. Section 2.3 describes innovation behaviors and welfare properties of the three allocation mechanisms. Section 2.4 characterizes the dynamic results with numerical exercises. For simplicity, sections 2.3 and 2.4 focus on the linear substitution case. Non-linear substitution is considered in Section 2.5 with the intuition carried from the previous analyses. Section 2.6 discusses some modeling issues. Finally, Section 2.7 concludes with some remarks.

2.2 Environment

This is an economy of discrete infinite-time horizon. In this environment, two firms X and Y can technically improve quality of their corresponding products x and y to serve a unit measure of identical consumers. Besides x and y, there is another product z acting as the numeraire.

2.2.1 Consumers

In each period, consumers are endowed with B units of the numeraire z. As all of the products are perishable, consumers make static decisions to maximize the one-period utility

$$\max_{x,y,z\geq 0} \left\{ u\left[(\theta_x x)^\alpha + (\theta_y y)^\alpha\right] + z \right\}, \ \alpha \in (0,1]$$
$$\text{s.t. } p_x x + p_y y + z = B, \tag{2.1}$$

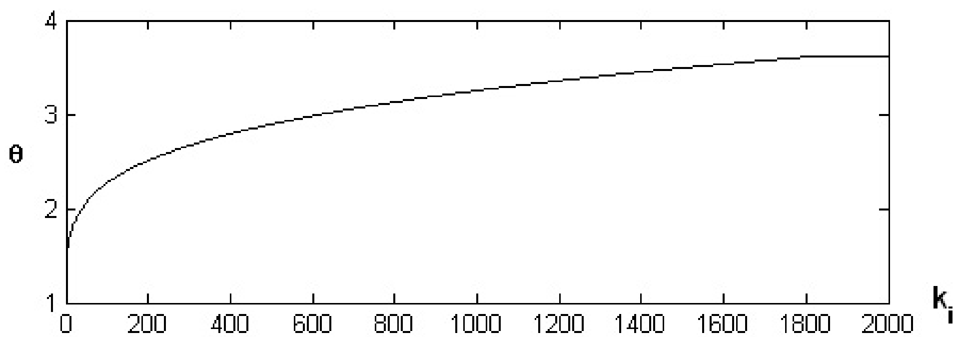
where x, y, z are quantities of consumption; θ_x and θ_y are *quality indices*; α is the substitution parameter; and u is a strictly concave function, specifically it takes the constant relative risk aversion (CRRA) form with $r_R = \sigma \in [\frac{1}{2}, 1)$. There are several highlighted features. First, as α ranges from above 0 to 1, products x and y become increasingly closer substitutes. Second, the concavity of function $u(\cdot)$ is necessary to accommodate the linear substitution case in which $\alpha = 1$. The reason why $u(\cdot)$ takes this CRRA class is to produce unambiguous effects and will be explained later. Third, budget B is assumed to be large enough so that $z > 0$ always holds in equilibrium. This condition guarantees that a monopoly firm will never charge an infinite price. Fourth, in this specification, the effective consumption quantity is the product of physical quantity and the corresponding quality. In addition, quality is also subject to the law of diminishing marginal utilities.

As consumers maximize their utility every period without any intertemporal choice, they will effectively have the maximized life-time utility. The discount factor is $\beta \in (0, 1)$, which is also the discount factor for firms.

2.2.2 Firms

Production has two dimensions: quality and quantity, which will be specified in the corresponding order. First, firms X and Y are respectively characterized by the *know-how stocks* k_x and k_y. A larger know-how stock embodied in a superior blueprint bears the notion of *technological progress.* A know-how stock k_i for $i \in \{x, y\}$ is related to its corresponding quality index θ_i by a common *valuation function* $\theta(\cdot)$. Specifically $\theta_i = \theta(k_i)$ where $\theta(0) = 1$, $\theta'(\cdot) \geq 0$, $\theta''(\cdot) \leq 0$, and $\lim_{k_i \to \infty} \theta'(k_i) = 0$. A specific valuation function is illustrated by Figure 2.1.

Figure 2.1: Valuation function $\theta(k_i)$



In words, with a larger know-how stock, a firm can produce a new product generation which is more appreciated by consumers. For simplicity, we assume that there is a threshold know-how stock k^* beyond which consumers do not see a difference in quality, i.e. $\theta(k_i) = \theta^* \ \forall k_i \geq k^*$.

Product supremacy is tied to the ordering of θ_x and θ_y, e.g. firm X has the supremacy if $\theta_x > \theta_y$, or equivalently $k_x > k_y$ for $k_x, k_y < k^*$. Firms can spend on R&D to accumulate more know-how. Let $\lambda_i \in [0, 1]$ be the choice variable for $i \in \{x, y\}$. The evolution function of a know-how stock is

$$k_i' = \begin{bmatrix} k_i + s(\lambda_i) & \text{with probability } \lambda_i \\ k_i & \text{with probability } 1 - \lambda_i, \end{bmatrix} \tag{2.2}$$

where ($\prime$) reads as next period only for state variables; $\lambda_i \in [0, 1]$ is the chosen *investment intensity* or *success rate*; and $s(\cdot)$ is the *innovation step function* with $s(0) = 0$, $s'(\lambda_i) > 0$, and $s''(\lambda_i) \leq 0$. As $s(\cdot)$ is continuous and defined on $[0, 1]$, it is bounded from above by the constant $\overline{s} = s(1)$. It is costly to carry out R&D projects. Let $c(\lambda_i)$ be the *innovation cost function* where

$c : [0, 1] \longrightarrow [0, \infty)$, $c'(\cdot) \geq 0$, $c''(\cdot) > 0$, i.e. it is too costly to have success for sure. It is noted that the linear accumulation technology in (2.2) does not change over time, while the curvature of $\theta(\cdot)$ does vary and will govern investment behaviors. Thus, to support the notion of a fixed technological structure, firms are not modeled to directly choose their quality indices.

Different from the quality ladder, a continuous step function implies continuous quality indices and equilibrium price ratios, which significantly enrich the set of competition strategies. Potentially, step and success rate can be modeled as separate choices. However, to keep the model and its computation tractable, the step is assumed to be a positive function of the success rate. In other words, firms decide on how far they want to progress next period and their efforts are subject to some uncertainty. Effectively, the expected innovation step is an increasing function of the R&D effort which is normalized to live in $[0, 1]$. In addition, we restrict $s(\cdot)$ so that the expected step is convex in effort (Appendix A.1).

Finally, firms face no capacity constraints. In addition, it costs w units of the numeraire to produce one unit of either x or y. Unit cost w does not depend on the know-how stocks. Consequently, it is optimal for firms to produce with the most superior blueprint, i.e. at the highest quality level.

2.2.3 Timing and Equilibrium Concepts

Later analyses will deal with investment performance and welfare properties of a monopoly, a duopoly, and the social planner. The monopoly and social planner can control all activities of both firms. These three considerations share a general timing as follows.

First, at the beginning of each period, all agents observe quality levels. Second, firms decide on prices and the associated non-negative production volumes. In the duopoly case, firms compete in prices, i.e. they engage in a

Bertrand game. It is assumed that production and purchase take very small amount of time. Consequently, firms pay all factor costs at the beginning of the period after they collect revenue from consumers. Third, right after the revenue collections and productive factor payments, firms decide on a non-negative investment intensity for potential R&D projects. Firms can finance R&D expenses from their profits or borrowings. Assume that firms can borrow up to the amount they want at the market interest rate $r = 1/\beta - 1$, and they borrow only for investments. In addition, principal and interest payments are enforced so that if a firm does not pay back it will suffer some money-equivalent punishment. For example, if the firm borrows c to pay for R&D activities, the discounted future payments count $-c$ to the present value. These assumptions mean that there are no differences between self-financing and borrowing, and firms have no effective budget constraints. An investment of $c > 0$ is desirable only if the expected present value of future gains is strictly greater than c. If a firm decides to invest, it will choose the optimal intensity to maximize the β-discounted sum of profit flows. In the duopoly setting, firms follow some simultaneous Markovian strategies. Finally, the outcome of any R&D project is realized at the beginning of the next period, either a success or a failure.

In this setup, innovation investments depend on know-how stocks or quality levels–the state, and pricing does not affect the evolution of the state. Thus, price decisions (static) do not affect investment strategies (dynamic).

2.3 Linear Substitution and Behaviors

For expositional simplicity, especially when dealing with welfare, we take linear substitution, i.e. $\alpha = 1$, as the base case and carry out necessary analyses in this section and the next one. As products are linearly substituted, it is later shown that generically only one product, either x or y, is produced and consumed every period. Simultaneous consumption only occurs when x

and y have the same quality levels under duopoly. In Section 2.5, we will look at the non-linear substitution case where $\alpha \in (0, 1)$.

2.3.1 Utility Maximization

With a quasi-linear preference, consumers maximize their utility by first choosing the budget share $b = B - z$ and then spend this amount on products x and y. Given that money is optimally spent on x and y, b is chosen at the point where its marginal utility is one.

Let's look at how consumers divide b between the two competing products. Given prices, consumers solve the following sub-problem

$$\max_{x,y\geq 0} u(\theta_x x + \theta_y y) \text{ s.t. } p_x x + p_y y = b. \tag{2.3}$$

Define p_x/θ_x and p_y/θ_y as quality-adjusted prices (QAP). In this linear substitution case, consumers only buy from the firm who offers a lower QAP. When the two QAPs are equal, we assume that consumers demand $x = y$. Details of these two cases are as follows.

First, if only product x is consumed (without loss of generality) and hence $x = b/p_x$, $b \in (0, B)$ is chosen to maximize $u(\theta_x b/p_x) + B - b$ with the first-order condition (FOC) $(\theta_x/p_x)u'(\theta_x b^*/p_x) = 1$. The optimal allocation is

$$\{x^*, y^*, z^*\} = \{b^*/p_x, 0, B - b^*\}. \tag{2.4}$$

Second, if $p_x/\theta_x = p_y/\theta_y = (p_x+p_y)/(\theta_x+\theta_y)$, by the above assumption, $x = y = b/(p_x + p_y)$. The budget share b is chosen to maximize $u((\theta_x + \theta_y)b/(p_x + p_y)) + B - b$, or equivalently $u(\theta_x b/p_x) + B - b$. The FOC is the same as in the first case. Let b^{**} be the solution, the desirable consumption bundle is

$$\{x^{**}, y^{**}, z^{**}\} = \{b^{**}/\left(p_x + p_y\right), b^{**}/\left(p_x + p_y\right), B - b^{**}\}. \tag{2.5}$$

Proposition 2.3.1. *With the assumptions on consumer utility, in the two previous cases, $b(\theta_x, p_x)$ increases in θ_x (and hence k_x) and decreases in p_x. In other words, consumers spend more on the innovation products if either quality is higher or price is lower, and vice versa. Explicitly, $b(\theta_x, p_x) = (\theta_x/p_x)^{(1-\sigma)/\sigma}$.*

Proof. This is an application of the implicit function theorem, based on initial assumptions of the utility function (Appendix A.2). ■

In the following analyses, this consumption behavior will be taken into account by the agents in different market structures. There is a general observation that, as unit costs are independent of quality, agents only consider producing products with their latest generations.

2.3.2 Monopoly Pricing and Investment

The monopoly effectively controls two firms and form a perfect cartel, as like that in Ericson & Pakes (1995). Starting with a zero know-how stock and a normalized quality indices $\{1, 1\}$, a monopoly maximizes its discounted infinite life-time profit by deciding on pricing and R&D investment every period.

Static Pricing. The first observation is that, in every period, the monopoly only produces and sells one product. Proposition 2.3.1 shows that revenue is increasing in quality. In addition, unit costs of x and y are the same. Thus, the monopoly only commercializes either the product with quality supremacy, or assumably x in the case of equal quality levels. Assume now that x is the chosen product as its quality is at least as high as that of y. The monopoly chooses a price to maximize one-period profit, i.e. $\max_{p_x}\{x(p_x - w)\}$, or equivalently

$$\max_{p_x \geq w} \left\{ b(p_x) \left(1 - \frac{w}{p_x}\right) \right\}. \tag{2.6}$$

The first term is a decreasing function of price, while the second is increasing in price. In fact, the profit maximization problem is well defined (Appendix

A.3), and the optimal price is

$$p_x = \frac{w}{1-\sigma}. \tag{2.7}$$

Next, the monopoly one-period profit function takes the closed form

$$\Pi^M(k_x) = \sigma \left(\frac{(1-\sigma)\,\theta_x}{w} \right)^{(1-\sigma)/\sigma}. \tag{2.8}$$

It is noted that the price function does not depend on quality, which comes from the quasi-linear form. Whereas, the profit function is increasing in θ_k or k_x. In addition, for $\sigma \geq 1/2$, it is straightforward to show that the first-order derivative $\Pi_1^M(k_x)$ is decreasing in k_x. Thus the profit function $\Pi^M(k_x)$ is concave in k_x.

Dynamic Investment. The second observation is that the monopoly wants to innovate only one product line, without loss of generality x, from the beginning. As noted earlier, the R&D technology, i.e. innovation step and cost functions, does not depend on time and state $\{k_x, k_y\}$. In addition, only the better blueprint $\max\{k_x, k_y\}$ matters to profit flows. Specifically, better quality can generate a larger profit as in (2.8). The argument runs as follows. Given any state $\{k_x, k_y\}$, the monopoly considers allocating a total cost of c on innovations to expectedly further $\max\{k_x, k_y\}$ the most in the next period. Recall that the expected step function $\lambda s(\lambda)$ is strictly increasing and convex in the effort. Consequently, it is the most beneficial to spend all of the effort on the technology edge. In particular, when $k_x = k_y \geq 0$, it is optimal for the monopoly to innovate only x.

We have just claimed that the monopoly innovates only x from the beginning. Given the reduced state k_x, the Bellman equation is

$$V^M(k_x) = \max_{\lambda_x \geq 0} \left\{ \Pi^M(k_x) - c(\lambda_x) + \beta E_{\lambda_x} V^M(k'_x) \right\}, \tag{2.9}$$

where $E_{\lambda_x} V^M(k'_x) = \lambda_x V^M(k_x^+) + (1-\lambda_x) V^M(k_x)$, and $k_x^+ = k_x + s(\lambda_x)$ when the R&D project succeeds. The Euler equation is (the subscripts denote derivatives)

$$-c'(\lambda_x) + \beta\left[\left(V^M\left(k_x^+\right) - V^M(k_x)\right) + \lambda_x \Pi_1^M\left(k_x^+\right) s'(\lambda_x)\right] \leq 0, \qquad (2.10)$$

where equality holds if $\lambda_x > 0$. It is observed that, by the envelop theorem, $V^M(k_x)$ inherits the concavity from $\Pi^M(k_x)$.

Equation (2.9) specifies a standard dynamic programming problem which has a unique solution. Far into the future, if at $k_x \geq k^*$, the optimal investment intensity is apparently zero, and $\{\Pi^M(k^*), V^M(k^*)\}$ are well specified. Thus, by backward induction the monopoly can solve for the entire time path of its optimal innovation efforts.

There exists a know-how level $k_M^* \leq k^*$ on and beyond which the monopoly does not want to invest. Clearly, at k^* and beyond, the second term of the LHS of (2.10) is virtually zero and the monopoly has no incentive to progress. Even though $\lim_{\lambda_x \to 0} c'(\lambda_x) = 0$, if $V_1^M(k_x)$ and $\Pi_1^M(k_x^+)$ tend towards 0 fast enough, the monopoly stops innovations before reaching k^*. In addition, we observe that

Proposition 2.3.2. *In the monopoly structure, for the range of k_x where $\lambda_x > 0$, the optimal investment intensity is decreasing in k_x.*

Proof. This is an application of the implicit function theorem, based on the concavity of the profit function and the second-order condition (SOC) of the Bellman equation (Appendix A.4). ■

Social Welfare. With a quasi-linear utility form, social welfare is the sum of consumer utility and monopoly profit in numeraire units. At each state k_x, consumer utility in equilibrium and the flow of social welfare respectively are $U^M(k_x) = u(\theta_x b/p_x) + B - b$ and $\Phi^M(k_x) = U^M(k_x) + \Pi^M(k_x) - c(\lambda_x)$. Specifically, they take the closed forms

$$U^M(k_x) = \frac{\sigma}{1-\sigma}\left[\frac{(1-\sigma)\,\theta_x}{w}\right]^{(1-\sigma)/\sigma} + B, \tag{2.11}$$

$$\Phi^M(k_x) = \frac{\sigma(2-\sigma)}{1-\sigma}\left[\frac{(1-\sigma)\,\theta_x}{w}\right]^{(1-\sigma)/\sigma} - c(\lambda_x) + B. \tag{2.12}$$

Thus, the discounted life-time social welfare with a monopoly structure is defined recursively as follows

$$W^M(k_x) = \frac{1}{1-\beta(1-\lambda_x)}\left\{\Phi^M(k_x) + \beta\lambda_x W^M\left(k_x^+\right)\right\}, \tag{2.13}$$

where $\lambda_x = \lambda_x(k_x) \geq 0$ and $k_x^+ = k_x + s(\lambda_x(k_x))$. It is easy to find the maximal long-run social welfare $W^M(k_m^*)$ which is

$$W^M(k_m^*) = \frac{\Phi^M(k_m^*)}{1-\beta}. \tag{2.14}$$

Based on (2.13) and (2.14), we can establish the time path of life-time social welfare backward from the maximal know-how stock k_m^*, and calculate the *ex ante* value $W^M(0)$.

2.3.3 Duopoly Pricing and Investment

For tractability, no entry and exit are allowed. In particular, a firm making zero profit can stay forever in the market. The two firms compete to gain market share in each period and race to have product supremacy.

Static Pricing. As noted earlier, price decisions do not have effects on investments. Thus, both firms will charge a price no less than unit cost, i.e. $p_x \geq w$ and $p_y \geq w$. If quality levels are equal, the duopoly firms play a standard Bertrand game in which equilibrium prices are w, each firm produces half of the quantity demanded, and both make zero profits.

The case of different quality levels is more interesting. Knowing consumer behavior and conditional on quality levels, each firm wants to monopolize the market by choosing a price which constitutes an infinitesimally lower QAP. This competition behavior is rational because the price effect is very small while the market share effect is very large. As X and Y try to cut down each other in QAP bit by bit, the laggard will hit the lower bound w first, and hence the front-runner with product supremacy has the advantage in pricing. Specifically, the front-runner will choose a price such that its QAP is ε less than that of the laggard at the lower bound. We assume that the equilibrium market share holds at the limit as $\varepsilon \to 0$. In addition, the laggard produces no output and potentially charges a price equal unit cost w in equilibrium.

If X is the front-runner, the firm monopolizes the market by confining its price such that

$$w \leq p_x \leq \frac{\theta_x}{\theta_y} w. \tag{2.15}$$

This strategic monopoly will then pick the price that maximizes its one-period profit. The earlier analysis shows that the profit function in price of an absolute monopoly has a single peak. The strategic monopoly puts the constraints in (2.15) on the domain of that profit function and can easily see the optimal price as follows

$$p_x = \left[\begin{array}{ll} w/(1-\sigma) & \text{if } 1/(1-\sigma) \leq \theta_x/\theta_y \\ (\theta_x/\theta_y)\, w & \text{if } 1/(1-\sigma) > \theta_x/\theta_y. \end{array} \right. \tag{2.16}$$

Recall that $\theta_x = \theta(k_x)$ and $\theta_y = \theta(k_y)$. Plugging the optimal price in (2.16) into the profit function in (2.6), we have the profit function of the front-runner. In the first case, i.e. $1/(1-\sigma) \leq \theta_x/\theta_y$, it is shown earlier that the profit function is increasing in k_x. In the second case, i.e. $1/(1-\sigma) > \theta_x/\theta_y$, price p_x is increasing in θ_x and decreasing in θ_y. Consequently, as the price is on the increasing side of a single-peaked profit function, equilibrium profit of a front-runner is increasing in k_x and decreasing in k_y. In this second case, the expression for the front-runner X's profit is

$$\Pi^D(\omega)_{\{1/(1-\sigma)>\theta_x/\theta_y\}} = \left(\frac{\theta_y}{w}\right)^{(1-\sigma)/\sigma}\left(1-\frac{\theta_y}{\theta_x}\right). \tag{2.17}$$

where $\omega = (k_x, k_y)$. In combination of the equal and unequal quality cases, let $\Pi^D(\omega)$, be the profit function of firm X. Observe that $\Pi^D(\omega) = 0$ if $k_x = k_y$, and $\Pi^D(\omega) > 0$ if $k_x > k_y$. Thus, total profit of the duopoly is either zero or equal the profit of the front-runner.

Dynamic Investment. As mentioned earlier, we look for MPE of the game. Markov strategies are strategies that depend only on payoff-relevant information of the history up to the current period. Maskin & Tirole (2001) define MPE to be equilibria in Markov strategies. In our problem, the payoff-relevant information in each period is the state $\omega = (k_x, k_y)$. The firms base exclusively on this information set to play and not on how they reach that information set. Specifically, given ω and λ_y, the Bellman equation for X is

$$V^D(\omega) = \max_{\lambda_x \geq 0}\left\{\Pi^D(\omega) - c(\lambda_x) + \beta E_{(\lambda_x,\lambda_y)} V^D(\omega')\right\}, \tag{2.18}$$

where $\omega' = (k'_x, k'_y)$, and for $k_x^+ = k_x + s(\lambda_x)$, $k_y^+ = k_y + s(\lambda_y)$,

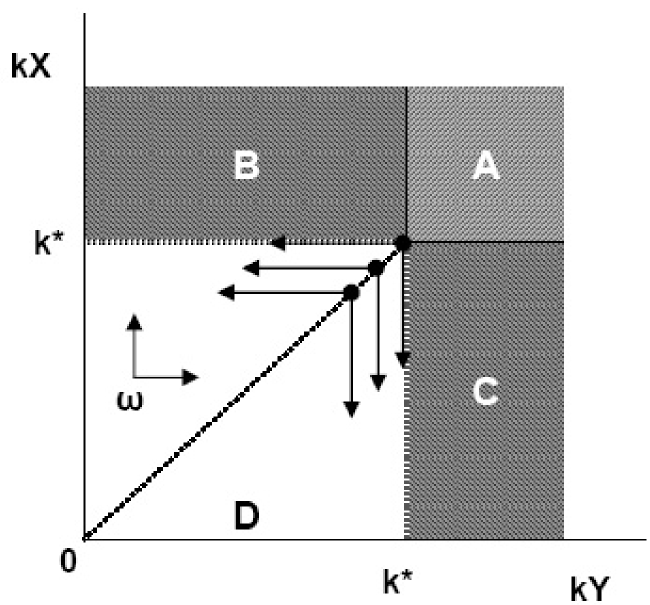
$$\omega' = \begin{cases} \left(k_x^+, k_y^+\right) & \text{with probability } \lambda_x\lambda_y \\ \left(k_x^+, k_y\right) & \text{with } \lambda_x(1-\lambda_y) \\ \left(k_x, k_y^+\right) & \text{with } (1-\lambda_x)\lambda_y \\ \left(k_x, k_y\right) & \text{with } (1-\lambda_x)(1-\lambda_y). \end{cases}$$

By the same token, given ω and λ_x, Y also has the same Bellman equation as in (2.18) with necessary changes in the index labels.

Definition 2.3.3. *A symmetric MPE in pure strategies of the duopoly R&D game is the investment function $\lambda(\cdot)$, which is associated with the discounted life-time profit $V^D(\cdot)$, such that for any state $\omega = (k_x, k_y) \in \Omega \subset \mathbf{R}^2$: given that firm Y follows the policy rule $\lambda(\cdot)$, firm X finds $\lambda(\cdot)$ as the optimal decisions for the problem in (2.18); and vice versa, given that X plays $\lambda(\cdot)$, Y also finds it optimal to follow $\lambda(\cdot)$.*

We are interested in the existence, uniqueness, and characterizations of the possible equilibria. Existence and uniqueness of MPE are discussed in Maskin & Tirole (2001) and Doraszelski & Satterthwaite (2007). In the current environment, the policy function $\lambda(\cdot)$ is bounded by construction. Thus, we expect that an MPE in pure strategies exists. The current setup does not support closed-form solutions to the R&D game. However, we can address existence and uniqueness in a special way. It is noted that an MPE must satisfy the subgame perfection argument. The existence of the threshold k^* means that equilibrium behaviors on and beyond a set of nodes, which lie on all possible paths of the game, can be constructed. Behaviors in earlier nodes can then be solved by backward induction.

Figure 2.2: Duopoly: backward induction



Specifically, the backward induction argument is illustrated by Figure 2.2, which describes a 2-dimensional state space of know-how stocks. From each point $\omega = (k_x, k_y)$, the two firms consider moving upwards and to the right. Recall that beyond k^* firms cannot raise their quality indices. The state space is partitioned into areas A, B, C, and D, in which we need to solve for the decision rule and value function.

First, in areas A, B, and C, we know that both firms have no incentive to invest at all, and their value function is $\Pi^D(\omega)/(1-\beta)$. More specifically, in area A both firms cannot raise consumers' valuation by accumulating more know-how and hence make zero profit. In area B, firm X does not invest because it is already beyond k^*, and firm Y—the laggard—does not want to make losses all the way to area A, where the expected profit is zero. By the same token, no firms invest in area C.

Second, equilibrium value and investment functions in area D are solved by backward induction, which is illustrated by the arrows. Without loss of generality, the reference firm is X. Let $\widetilde{\omega}$ be the permuted state of ω, i.e. $\widetilde{\omega} = (k_y, k_x)$. Starting at the top right corner, firm X goes horizontally for lower k_y before going vertically for lower k_x. Observe that at each point ω, future equilibrium investments and the associated value functions are known to X. However, X does not know its value function and the competitor's decision at the current state ω. By symmetry, the competitor's decision at ω is the same as firm X's decision at $\widetilde{\omega}$, which is currently unknown. That means X needs to find its optimal investments at ω and $\widetilde{\omega}$ simultaneously. For this reason, the state-dependent consideration for X is called the pairwise fixed point problem, which is a component of the entire R&D game. Based on (2.18) and the corresponding FOC (Appendix A.5), the best response functions of firm X with respect to firm Y's decisions at ω and $\widetilde{\omega}$ can be constructed. Thus, existence and uniqueness of the whole R&D game depend on the number of crossing points between these two best response functions, which vary across area D. Specifically, we have existence and uniqueness if and only if the single crossing property holds for each point on the state space. In the next section, we will implement this backward induction solution in a discrete game.

Social Welfare. At each state $\omega = (k_x, k_y)$, the flow of social welfare is $\Phi^D(\omega) = U^D(\omega) + \Pi^D(\omega) - c(\lambda_x) - c(\lambda_y)$, where $U^D(\omega)$ is consumers' one period utility. For $k_x \geq k_y$, $U^D(\omega) = u(\theta_x b/p_x) + B - b$. For $k_x < k_y$, $U^D(\omega) = u(\theta_y b/p_y) + B - b$. Recall that there are two cases of front-runner pricing depending on where θ_x/θ_y lies relative to $1/(1-\sigma)$. It is more interesting to focus on the second case where $\theta_x/\theta_y < 1/(1-\sigma)$ always holds. Under this assumption, the one-period utility when $k_x \geq k_y$ is

$$U^D(\omega) = \frac{\sigma}{1-\sigma}\left(\frac{\theta_y}{w}\right)^{(1-\sigma)/\sigma} + B, \tag{2.19}$$

and the flow of social welfare is

$$\Phi^D(\omega) = \left(\frac{1}{1-\sigma} - \frac{\theta_y}{\theta_x}\right)\left(\frac{\theta_y}{w}\right)^{(1-\sigma)/\sigma} - c\left(\lambda_x\right) - c\left(\lambda_y\right) + B. \tag{2.20}$$

Recursively, the discounted life-time social welfare function in ω is

$$W^D\left(\omega\right) = \Phi^D(\omega) + \beta E_{(\lambda_x,\lambda_y)} W^D\left(\omega'\right), \tag{2.21}$$

where ω' and the integration $E_{(\lambda_x,\lambda_y)}$ are defined in (2.18), and $\{\lambda_x, \lambda_y\}$ follow $\lambda(\cdot)$ which is an MPE in Definition 2.3.3. Social welfare value $W^D\left(\omega\right)$ for ω at points in areas A, B, and C (Figure 2.2) can trivially be specified. By backward induction, we can calculate the *ex ante* social welfare $W^D\left(0,0\right)$.

2.3.4 The Social Planner

The social planner's objective is to maximize consumers' discounted life-time utility by controlling firms' pricing and investment activities. As in the monopoly structure, the social planner needs to produce and innovate only one product line, assumably x. Given the quality θ_x, the social planner charges a price equal unit cost w in every period. The one-period utility function is $U^S(k_x) = u\left(\theta_x b/w\right) + B - b$, and explicitly

$$U^S(k_x) = \frac{\sigma}{1-\sigma}\left(\frac{\theta_x}{w}\right)^{(1-\sigma)/\sigma} + B. \tag{2.22}$$

In (2.22), $U^S(k_x)$ is increasing and concave in k_x. The social planner solves the following Bellman equation

$$V^S(k_x) = \max_{\lambda_x \geq 0} \left\{ U^S(k_x) - c(\lambda_x) + \beta E_{\lambda_x} V^S(k_x') \right\}, \tag{2.23}$$

where $E_{\lambda_x} V^S(k_x') = \lambda_x V^S(k_x^+) + (1 - \lambda_x) V^S(k_x)$. The FOC of (2.23) is

$$-c'(\lambda_x) + \beta \left[\left(V^S\left(k_x^+\right) - V^S(k_x) \right) + \lambda_x U_1^S\left(k_x^+\right) s'(\lambda_x) \right] \leq 0, \tag{2.24}$$

where equality holds for $\lambda_x > 0$. Again, by the envelop theorem, $V_1^S(k_x) = U_1^S(k_x) > 0$ and $V_{11}^S(k_x) = U_{11}^S(k_x) < 0$. Like in the monopoly, we have

Proposition 2.3.4. *Under the social planner, in the range of k_x such that $\lambda_x(k_x) > 0$, investment effort is decreasing in k_x, i.e. $\lambda_x'(k_x) < 0$.*

Proof. The argument follows the same line as in Proposition 2.3.2. ■

In this structure, observe that the discounted social welfare $W^S(k_x) = V^S(k_x)$. Let k_S^* be the threshold on and beyond which the social planner does not find it beneficial to innovate. The maximal long-run social welfare is $W^S(k_S^*)$. By backward induction, we can also find *ex ante* social value $W^S(0)$.

2.3.5 Comparisons of Social Welfare

Let a gross social welfare flow be the sum of utility and profits. Thus $\Phi^{GM}(k_x) = U^M(k_x) + \Pi^M(k_x)$; $\Phi^{GD}(k_x, k_y) = U^D(k_x) + \Pi^D(k_x)$ for $k_x \geq k_y$; and $\Phi^{GS}(k_x) = U^S(k_x) + \Pi^S(k_x)$. Respectively, these functions have the closed forms

$$\Phi^{GM}(k_x) = \left[\sigma (2 - \sigma)(1 - \sigma)^{1/\sigma - 2} \right] \left(\frac{\theta_x}{w} \right)^{(1-\sigma)/\sigma} + B, \tag{2.25}$$

$$\Phi^{GD}(k_x, k_y) = \left[\frac{1}{1 - \sigma} - \frac{\theta_y}{\theta_x} \right] \left(\frac{\theta_y}{w} \right)^{(1-\sigma)/\sigma} + B, \tag{2.26}$$

$$\Phi^{GS}(k_x) = \left[\frac{\sigma}{1 - \sigma} \right] \left(\frac{\theta_x}{w} \right)^{(1-\sigma)/\sigma} + B. \tag{2.27}$$

In comparison of the three market structures, there are some observations. First, $\Phi^{GS}(k_x) > \Phi^{GM}(k_x)$. This result comes from the fact that $(2-\sigma)(1-\sigma)^{(1-\sigma)/\sigma}$ is an increasing function ranging from 0.75 to near 1 for $\sigma \in [\frac{1}{2}, 1)$. Second, $\Phi^{GS}(k_x) \geq \Phi^{GD}(k_x, k_y)$. The reason is $\Phi^{GD}(k_x, k_y)$ is increasing k_y which is bounded by k_x. In addition, when $k_x = k_y$, $\Phi^{GS}(k_x) = \Phi^{GD}(k_x, k_y)$. Third, $\Phi^{GD}(k_x, k_y) > \Phi^{GM}(k_x)$. To see why, note that the ordering between the two is equivalent to the ordering of the following two expressions

$$\left[\frac{1}{1-\sigma} - \frac{\theta_y}{\theta_x}\right] \left(\frac{\theta_y}{\theta_x}\right)^{(1-\sigma)/\sigma},$$

which is increasing in $\theta_y/\theta_x \in [1/\theta^*, 1]$, and

$$\sigma\,(2-\sigma)\,(1-\sigma)^{1/\sigma-2}.$$

Recall that θ_x/θ_y is bounded from above by $1/(1-\sigma)$, i.e. $\theta^* < 1/(1-\sigma)$, which implies $1/\theta^* > (1-\sigma)$. When $\theta_y/\theta_x = (1-\sigma)$, the two expressions are equal. Thus, in the range $[1/\theta^*, 1]$, the former is strictly greater than the latter. Fourth, by the same token, it is straightforward to show $U^S(k_x) \geq U^D(k_x) > U^M(k_x)$, where $U^S(k_x) = U^D(k_x)$ for $k_x = k_y$. In combination, we have established

Lemma 2.3.5. *For $\theta^* < 1/(1-\sigma)$, and $k_x \geq k_y$, $\Phi^{GS}(k_x) \geq \Phi^{GD}(k_x, k_y) > \Phi^{GM}(k_x)$, which means gross social welfare flow is the largest under the social planner and smallest under monopoly. In addition, utility components of these functions also have the similar ordering, i.e. $U^S(k_x) \geq U^D(k_x) > U^M(k_x)$.*

Proposition 2.3.6. *The social planner dominates monopoly in both dynamic social welfare and R&D efforts, i.e. $W^S(k_x) > W^M(k_x)$ and $\lambda^S(k_x) \geq \lambda^M(k_x)$ $\forall k_x$. In addition, the social planner dominates duopoly in social welfare, i.e. $W^S(k_x) \geq W^D(k_x, k_y)$ $\forall k_x \geq k_y$.*

Proof. Appendix A.6. ■

At this point, we do not have analytical solution to the duopoly problem. Hence, further comparison results need to rely on different numerical exercises in the next sections.

2.4 Numerical Characterizations

Based on specific parametrization, this section further characterizes investment behaviors and welfare properties of the three allocation mechanisms, i.e. a monopoly, a duopoly, and the social planner, in the linear substitution case ($\alpha = 1$). As the focus is theoretical analyses, we do not attempt to calibrate the model to any specific markets. The benchmark values of the parameters are summarized in Table 2.1.

Table 2.1: Benchmark parameter values

Description	Specification
Curvature of CRRA utility	$\sigma = 0.8$
Curvature of CES sub-utility	$\alpha = 1$
Interest rate	$r = 5\%$
Intertemporal discount factor	$\beta = 0.952$
Consumer budget	$B = 0$
Production cost	$w = 1$
Know-how space	$k \in \{0, 1, .., 2000\}$, $k^* = 1800$
Valuation function	$\theta(k) = \gamma k^\delta + 1$, $\gamma = 0.4, \delta = 0.25$
Choice of success rate	$\lambda \in \{0, .01, .02, .., 1\}$
Innovation step function	$s(\lambda) = \psi\lambda$, $\psi = 100$
Innovation cost function	$c(\lambda) = \kappa \frac{\lambda}{(1-\lambda)^\eta}$, $\kappa = 0.02, \eta = 5$

There are some notes on the choice of parameter values. First, the CRRA utility function is concave, i.e. $\sigma < 1$. This condition means consumers will demand more of a product if its quality is increasing. In addition, $\sigma \geq \frac{1}{2}$ holds to guarantee that one-period profit functions are concave. Second, the interest rate is assumed to be at the annual level $r = 5\%$ which is often used by the literature. The discount factor then follows $\beta = 1/(1 + r)$.

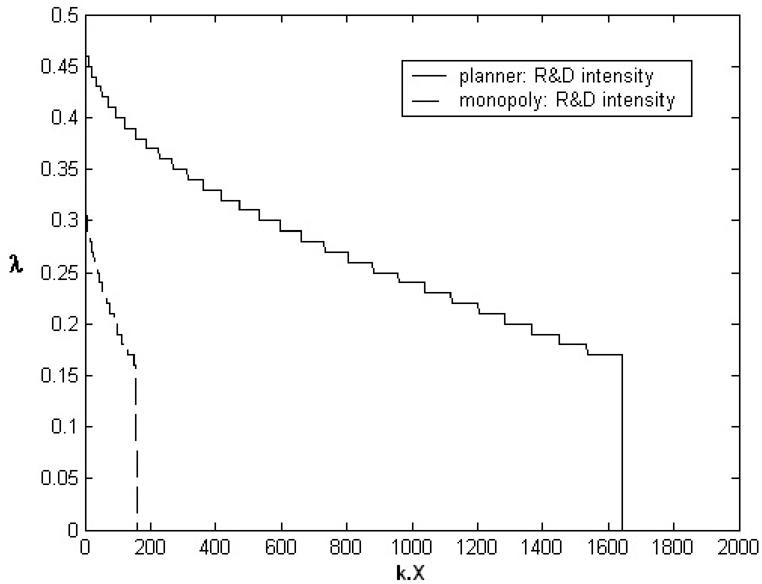
It is noted that period length is not necessarily one year. Third, the budget constraint B does not play any role at this point and is normalized to be zero. Fourth, production cost w is set at 1 for simplicity. Fifth, the know-how space is composed of integers in $[0, 2000]$, and hence the state space Ω is a discrete grid. Sixth, the curvature of the valuation function lies in δ, and (γ, δ) are chosen to guarantee $\theta^* < 1/(1-\sigma)$. In addition, the valuation function is normalized so that the lowest quality is 1. Seventh, for compatibility, the step function maps each λ to exactly a point on the know-how space, and the parameter ψ governs the step sizes. Eighth, in the innovation cost function, η determines the curvature and κ plays the role of a scale. Finally, given the benchmark know-how space, the three characteristic functions, i.e. valuation, innovation step, and innovation cost, are constructed so that firms stop innovations relatively long before reaching k^*. The reason is that if it is optimal to innovate in the proximity of k^*, the investment functions in that proximity become bumpy and look like waves bouncing from a seawall. Clearly, this phenomenon comes from the abrupt change in the valuation curvature at k^*. Alternatively, given the characteristic functions, we can expand the state space to avoid this phenomenon. However, a larger state space means a heavier computational burden, especially in the duopoly case.

2.4.1 Social Planner vs. Monopoly

We begin with the comparison between the social planner and monopoly in terms of innovation efforts and social welfare. The comparison is simplified as these allocation mechanisms evolve effectively in one dimension (k_x).

R&D efforts and social welfare are presented in Figures 2.3 and 2.4. There are some major results in this comparison. First, as shown earlier, both the social planner and monopoly reduce their innovation efforts over the know-how space, and eventually hit some points beyond which no further

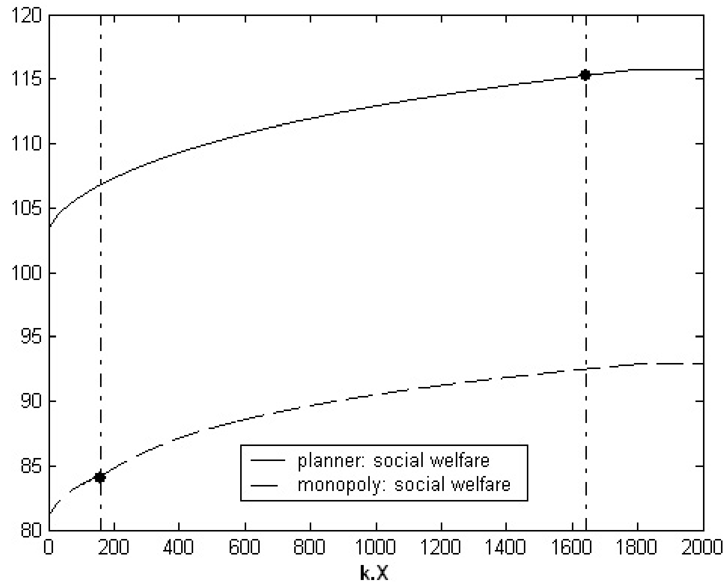
Figure 2.3: Social planner vs. monopoly: R&D intensity



Note: $\alpha = 1$; social planner investment follows (2.24); monopoly investment follows (2.10). Planner always makes more R&D effort than monopoly.

investments are beneficial. Second, the social planner always exerts more effort on innovations and reaches a higher maximal know-how stock than the monopoly. It is noted that the social planner makes innovation decisions based on a flow function larger and steeper than monopoly one-period profit. Third, though with more expenditures on R&D, the social planner generates higher life-time social welfare than the monopoly. The reason is that the net social welfare flows are higher under the social planner. In addition, the social planner has more chance to succeed in R&D projects and hence more effectively avoids wasteful investments than the monopoly. Thus, at the beginning, i.e. $k_x = 0$, *ex ante* social welfare under the social planner is larger than that under monopoly. Moreover, in the long run, the economy reaches a higher steady-state quality level and social welfare under the former than under the

Figure 2.4: Social planner vs. monopoly: social welfare



Note: $\alpha = 1$; planner welfare follows (2.23); monopoly welfare follows (2.13); the dots mark the maximal social welfare levels.

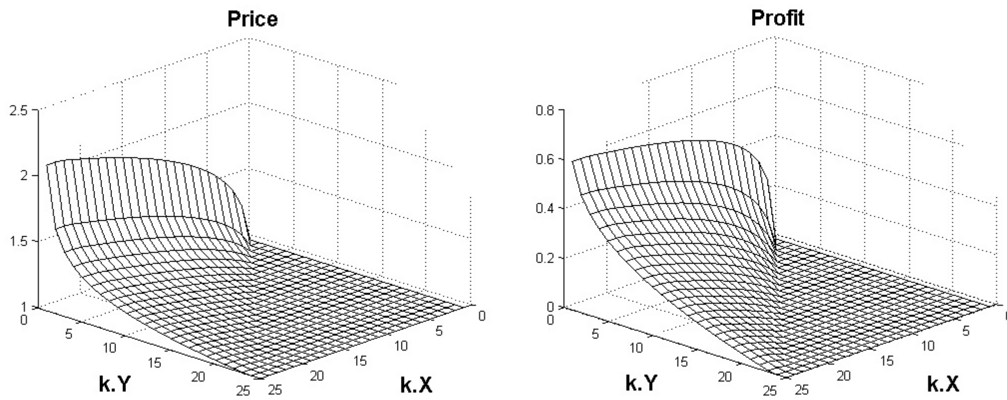
latter (the dots in Figure 2.4).

2.4.2 Duopoly Behavior

Static Pricing. In the linear substitution case, product supremacy is all that matters. Figure 2.5 illustrates the pricing behavior and profit of firm X. As a laggard, X ties its price at w and makes a zero profit. As a front-runner, the firm can charge a higher price which is subject to the quality ratio. Specifically, the front-runner's price and profit become higher if its relative quality increases. Note that the curvature of price and profit functions along k_x comes from consumers' valuation.

Dynamic Investment. As discussed earlier, solving for an MPE of the entire R&D game boils down to solving for a Nash equilibrium in every state $\omega \in \Omega$, following a backward induction fashion. Recall that given $\omega = (k_x, k_y)$,

Figure 2.5: Duopoly: price and profit functions (firm X)



(A) (B)

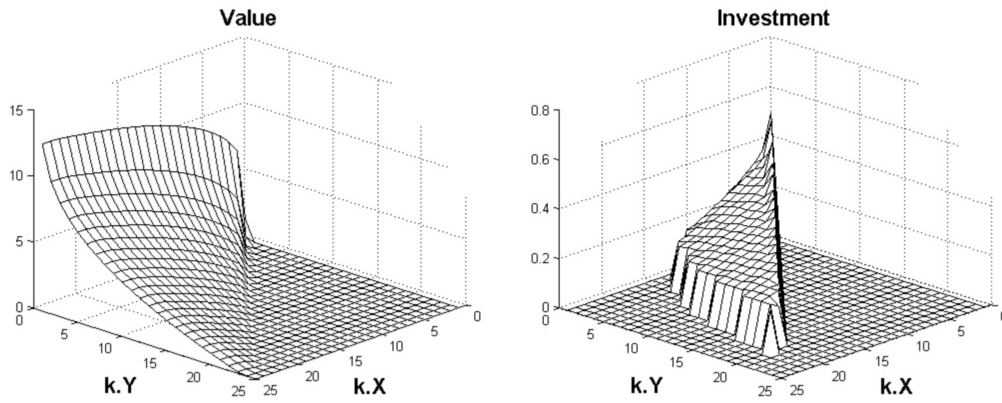
Note: $\alpha = 1$; $\theta^{*} < 1/(1-\sigma)$; price follows (2.16); profit follows (2.17). Higher quality levels lead to higher prices (A) and higher profits (B).

behaviors at weakly higher states $\{\omega'\}$ are already known, for $\omega' = (k'_x, k'_y)$, $k'_x \geq k_x$, $k'_y \geq k_y$, and at least one of the inequalities is strict. In addition, in every state ω, by symmetry, we need to find the fixed point of the R&D game between firm X and itself at the permuted state $\widetilde{\omega}$. In this game, players choose the R&D intensity λ in a compact set, i.e. $[0, 1]$.

The numerical algorithm for finding the fixed points is based on a simple interpretation of the Nash equilibrium concept. It runs as follows: given any state ω, construct the best response functions of the two firms and find the fixed point on a grid choice space. In implementation, most of the state games have exact fixed points. For the state games which do not have exact fixed points, we approximate the equilibrium with the closest grid point which is not Pareto dominated.

There are some distinguished features of the numerical equilibrium play. First, a firm may have expected positive value even when lagging in the market (Figure 2.6). The reason is that if the know-how gap is not too large, the lag-

Figure 2.6: Duopoly: value and investment functions (firm X)

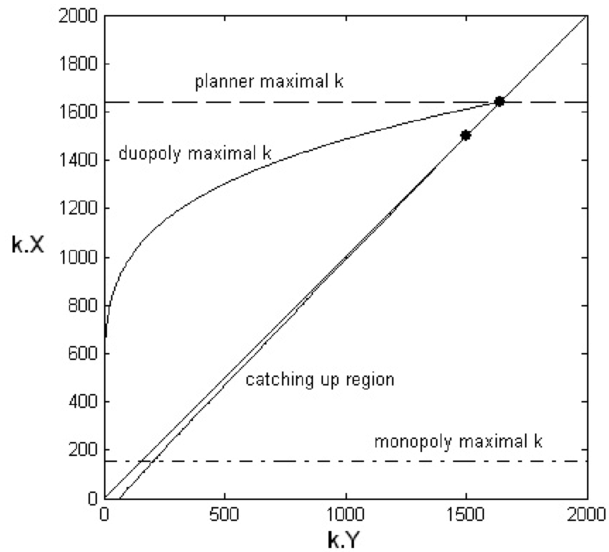


(A) (B)

Note: $\alpha = 1$; value follows (2.18); investment follows Definition 2.3.3. In (A), higher firm values come from higher quality levels. In (B), X only invests in a bounded region; R&D efforts are decreasing in k_x; competition escalades in the diagonal region.

gard has some chance to catch up with the front-runner via R&D investments. Figure 2.7 shows the region where the laggard still wants to catch up. Second, if too far behind, the laggard does not invest in R&D. Third, the R&D race is the most intensified when firms are close to each other, especially when they have equal know-how stocks. In other words, firms really want to break the balance to have an advantage in pricing. By comparing Figures 2.3 and 2.6 (panel B), given $k_x = k_y$ which are both low, a duopoly firm invests more than the social planner at k_x. Forth, even though the front-runner can leave the laggard so far behind that the laggard will never invest in R&D, the front-runner still has incentives to invest as it can raise the quality ratio for larger profits. Fifth, investment incentive generally decreases in know-how stock, reflecting the decreasing marginal valuation of consumers. When consumers appreciate quality improvements the most, i.e. for low know-how stocks or *fresh market*, firms invest intensively. When increases in know-how stock cannot raise much

Figure 2.7: Duopoly: investment boundaries (firm X)



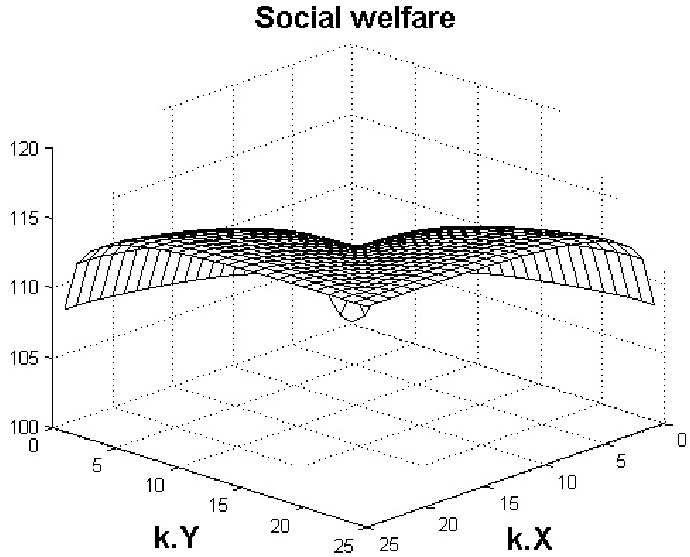
Note: $\alpha = 1$; upper curve specifies $\{\text{maximal } k_x\}$; lower curve & diagonal make up the catching up region. Duopoly technology frontier lies between those of the social planner and monopoly.

consumers' valuation, i.e. the market becomes mature, no firms invest in R&D. Specifically, Figure 2.7 illustrates the boundary beyond which a front-runner no longer wants to innovate. It can be seen that if the competitor ends up with low quality level, firm X does not have much incentives to progress far. However, if the two firms make relatively equal progress, they may push the frontier close to the social planner's maximal know-how stock.

Social Welfare. Figure 2.8 shows how the duopoly social welfare function looks like. It is symmetric with respect to the diagonal where the firms have equal product quality and consumers benefit the most. In addition, the boundary along which the society can effectively achieve maximal welfare value is defined by the technology frontier in Figure 2.7 (X is the front-runner).

Generally, the duopoly welfare function is increasing in k_x and k_y. However, it is cleaved a long the diagonal region. Given some k_y and in the neigh-

Figure 2.8: Duopoly: social welfare function



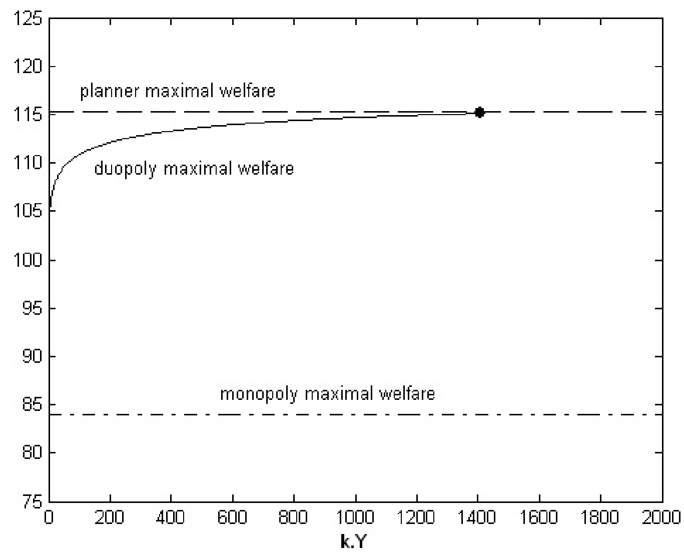
Note: $\alpha = 1$; social welfare follows (2.21). The function has a trench along the diagonal where firms make a lot of wasteful investments.

borhood where $k_x \leq k_y$, the social welfare function becomes flat. When k_x passes k_y, the function becomes steeper, and again increasing and concave in k_x. That is when neck-and-neck, firms intensify their competition effort and make a lot of wasteful investments from the point of view of the society.

2.4.3 Mechanisms: Innovation and Social Welfare

We compare the three allocation mechanisms here. First, Figure 2.7 shows an important result: the duopoly technology frontier is lower than that of the social planner and higher than that of the monopoly. This means the long-run social welfare of the mechanisms follows the same ordering (Figure 2.9). It is interesting that if the duopoly firms always advance together in equilibrium, they can drive social welfare to the level by the social planner. Second, the *ex ante* social welfare value is the highest under the social planner and lowest under monopoly. Specifically, the *ex ante* social welfare values under the social planner, a duopoly, and a monopoly respectively are $EWS = 102$, $EWD = 98$, and $EWM = 80$. Third, allocation mechanism does matter to the rates of technological progress. The monopoly is the slowest. The order between the social planner and a duopoly depends on where the firms are in the state space. A duopoly in the diagonal region advances more quickly than a social planner with the same latest technology.

Figure 2.9: Comparison of maximal welfare



Note: $\alpha = 1$; social planner and monopoly only innovate x and keep $k_y = 0$; duopoly technology frontier is that of front-runner X. Duopoly ranks lower than social planner and higher than monopoly in maximal social welfare.

In combination of the results: the social planner benefits the economy the most; a duopoly may generate an outcome in terms of technology and welfare comparable to that by the social planner, but not always; and the monopoly is definitely the least desirable. This welfare result means that more static and dynamic competition benefits the society in both the short run and long run.

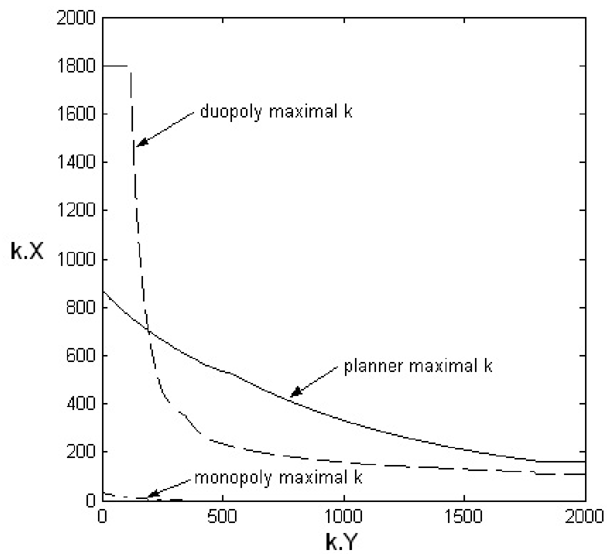
2.5 Nonlinear Substitution

We have analyzed investment behaviors and social welfare properties for the linear substitution case. In this section, the same considerations are extended to the nonlinear substitution case, where $\alpha \in (0, 1)$. In fact, analyses with nonlinear substitution are much more costly than those with linear substitution. We will present numerical results associated with $\alpha = 0.8$. We want to see if the previous key conclusions are invariant to product substitutability.

For the most part, the formulations of pricing and investment problems are similar to those in the linear substitution case (Appendix A.7). However, there are some major differences. First, consumers always want to consume both products x and y, making the demand function smoother. Thus, the monopoly and social planner will produce and innovate both products in every period. Two, the setup does not support closed form solutions to the static problems, especially the duopoly pricing game. These mean that we have to rely more on computations to characterize the nonlinear substitution case.

Before investigating investment behavior and social welfare, we need to understand the role of quality in profit maximization. Under monopoly, quality does not affect optimal prices for large B, which also holds for the linear substitution case. However, quality has the budget share effects, i.e. higher quality attracts more expenditure from consumers. This is the incentive for quality innovation under monopoly. Under duopoly, the market share for each firm and consumer expenditure are both tied to quality levels. In fact, a firm's one-period profit is increasing in its product quality and decreasing in that of the rival (Figure A.2, panel A). For this reason, firms are motivated to improve their own product quality. Different from the market power incentive, the sole interest of the social planner is raising consumers' utility through quality innovation. In this section, Figures 2.10 and 2.11 will provide the main comparisons. Further details about each allocation mechanism are presented in Appendix A.9.

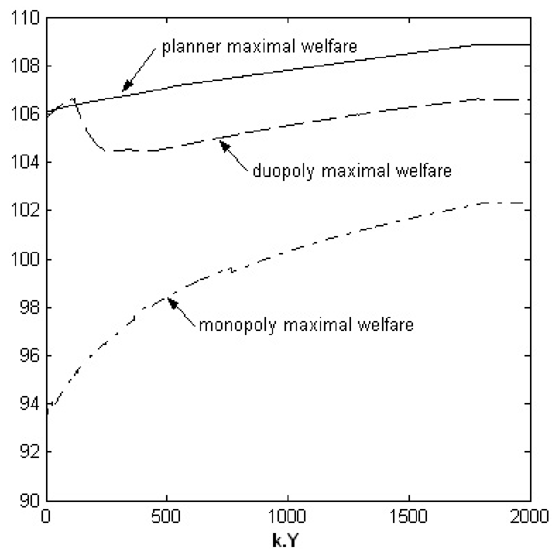
Figure 2.10: Technology frontiers (product x)



Note: $\alpha = 0.8$. The frontiers show the maximal know-how stocks beyond which firm X, under different market structures, will no longer innovate product x. The same holds for product y.

Figure 2.10 representatively shows the technology frontier of product x, beyond which no firms want to make R&D efforts. First, the monopoly's frontier is low while that of the social planner is much higher. However, the two mechanisms share the same pattern of decreasing innovation efforts along k_x (Appendix A.9). This common feature is intuitive for a concave valuation function. Interestingly, second, the duopoly frontier is not always lower than the planner counterpart. If the stochastic evolution in equilibrium is unbalanced, the duopoly may end up with a product with very high quality (even higher than under the social planner) and the other with low quality. The reason is that if the competitor is not lucky in its R&D projects and stays at low product quality, firm X will find it easy to raise its relative quality.

Figure 2.11: Maximal social welfare (x frontier)



Note: $\alpha = 0.8$. Maximal welfare values are associated with the technology frontiers in Figure 2.10.

As a sole innovator, neither a social planner nor a monopoly has incentives to progress in an unbalanced fashion, even though they may end up with unequal quality states as the economy evolves stochastically.

Thus the linear and nonlinear substitution cases differ greatly in duopoly innovation behavior. First, for a low k_y, firm X invests to improve its product longer with nonlinear substitution than with linear substitution. In addition, firm X's innovation intensity is increasing in k_y with linear substitution, while the reverse holds with nonlinear substitution. This difference is intuitive. In the linear substitution case, a firm too far behind abandons R&D efforts all together, and the front-runner can sustain its leadership without much effort. As the laggard lands at a higher quality level, the front-runner needs to innovate its product up to some optimal point, making the frontier increasing. In contrary, the laggard with nonlinear substitution can always make positive profits because consumers demand its product. Consequently, the laggard has greater incentives in raising its market share via quality innovation. Knowing this, the front-runner will need to make more efforts to have a greater lead. As the additional curvature for $\alpha < 1$ makes it more costly to raise relative quality, a firm's investment frontier is decreasing in product quality of its rival. For example, for a low k_y, firm X finds it easy to have greater lead and has a high frontier. However, for a higher k_y, it is not beneficial for firm X to go as far. When X is a laggard and k_y is increasing, the firm's marginal benefits from innovation is smaller and smaller. Second, R&D efforts are not intensified when firms are neck-and-neck with nonlinear substitution. The reason is that the laggard does not have to face a hazard of making zero future profits as in the linear substitution case.

Finally, the social welfare functions of all three market structures are increasing and concave in ω (Appendix A.9). As firms do not intensify investments when they are neck-and-neck, the social welfare function is not cleaved as in the linear substitution case. There are some main comparative results. First, welfare values conditional on states are unambiguously ordered. Specifically, given any state ω, it is always beneficial to switch from duopoly to the social planner or from monopoly to duopoly. This implies *ex ante* social welfare value is highest under the social planner and lowest under monopoly, which also holds in the linear substitution case. Second, maximal social welfare value depends on where the economy lands along the technology frontier (Figure 2.11). Specifically, monopoly always generates the smallest social welfare. In dynamics, the duopoly may generate higher welfare than the social planner. However, in the long run, the latter benefits the economy the most.

Though there are differences in terms of duopoly investment behavior, the linear and nonlinear substitution cases share the same policy message that more competitiveness benefits the economy.

2.6 Further Discussions

We consider some issues related to the choice of parameters and modeling in this section. First is how the results will change if we vary the key parameters regarding innovation step and cost functions. In general, varying the innovation technology does not affect the comparative results between different allocation mechanisms, given linear substitution or nonlinear substitution. In particular, if it is easier to innovate, i.e. the step function is higher or the cost function is lower, firms will make higher innovation efforts, given some state ω. It is noted that the continuous step function is implemented on a grid space. Thus, any change in the step function has to satisfy the condition that intensity decisions advance the state to exact grid points.

Second, we do not have to vary the valuation function to see how the model works, because its curvature varies with know-how stocks. Specifically, for each market structure and substitution degree, the key determinant behind innovation incentives is the curvature of the valuation function. For low know-how stocks, where the valuation function is steep, the marginal benefits of innovation is large and firms make great R&D efforts. In the long run, as the valuation function is flat, firms have small incentives in product improvements.

Third, the current choice of the threshold k^* is a technical assumption which makes the solution concepts more understandable and keeps the state space small enough for computational purposes. The key welfare results do not depend on the choice of k^*. Ideally, k^* should be chosen so that the corresponding slope of the valuation function, i.e. $\theta'(k^*)$, is smaller than any other magnitudes considered in R&D problems. For a large k^*, investment decisions are not subject to bouncing effects and look smoother. However, as k^* is increasing, the computational burden grows exponentially. Thus we have to make a trade off between smoothness and computational feasibility.

Fourth, we do not consider the dependence of production cost w on know-how stock. If unit cost is increasing in quality, we also expect that innovation intensity is decreasing. In empirical work, changes in production cost may be needed to make the model match with data. However, in this study, the assumption of an invariant unit cost is necessary to keep the model focus on quality innovation. In addition, real-life developments show that quality innovation is not necessarily associated with higher production cost. For simplicity, we also keep the innovation technology, i.e. innovation step and cost functions, independent of know-how stock.

Fifth, states are not allowed to move backwards. This assumption facilitates backward induction solutions and differentiates the current study with the existing literature, e.g. Pakes & McGuire (1994) and Doraszelski (2003). Clearly, the evolution rule, according to which a state ω follows, does influence firms' innovation incentives. In general, given the same effective state space Ω, the measure of ω such that some firm does not invest is larger with the nondecreasing state assumption. In addition, the long-run outcomes, especially in technology frontier, are different.

Finally, we restrict the game theoretic equilibrium concepts to Nash and MPE. Finer concepts which may rely on Folk's theorem are beyond the scope of our current interest. Those finer considerations demand much more elaborations and are left for future work.

2.7 Conclusion

Our main purpose is analyzing how different allocation mechanisms affect the quality innovation efforts and social welfare. The key technology structure is that firms can choose continuous innovation step to progress in the next period and state variables are non-decreasing. There are some main results. First, in the linear substitution case, the planner technology frontier is always superior to the counterparts under duopoly and monopoly. Second, in the nonlinear substitution case, a duopoly may follow an unbalanced evolution path and have a technology frontier not dominated by that under the social planner. Third, social welfare values are always highest under the social planner and lowest under monopoly.

We can extend the model in at least several ways. On one hand, some of them may not add much intuition to the current results. On the other, some potential extensions deserve separate research projects. Consequently, we keep the model as simple as possible to focus on the effects of allocation mechanism on innovation and welfare.

The analysis again advocates for the virtue of competition. Competition puts a downward pressure on prices and provides the incentives for firms to repeatedly expand the technology frontier, raising social welfare. At the same time, it should be borne in mind that intensified competition may lead to wasteful allocation of resources.

Chapter 3

Inference of Quality Innovation

3.1 Introduction

Relative prices are understood as conversion rates between commodity bundles of different countries or sectors. In standard growth and business-cycle models, relative prices vary due only to productivity shocks. *Ceteris paribus*, relative price and relative quantity, which is driven by relative productivity, should have a negative correlation. That is as we consume more of some product, marginal utility and hence relative price of that product will decrease. This negative correlation does not fully hold for the case of US services versus US goods from 1946-2006 (Figure 3.1): while services-goods relative quantity fluctuates, the relative price steadily increases over time, suggesting that the standard models may miss something important in the economy.

Potentially, there are other important sources of dynamics besides productivity. Among those is quality innovation–changes in utility level given the same consumption quantity. In the current study, we allow the coexistence of productivity shock and quality innovation, which are both exogenous. Given this coexistence, the study addresses three closely related questions: (i) with data on relative prices and potentially other variables, how can we separate the effects of productivity shock and quality innovation? (ii) given the separation, how are they individually and jointly characterized for the US, i.e. volatility, persistence, correlation and causation? and finally (iii) what are the implications of the separation exercise to a certain set of business-cycle and growth models?

The study shows that based on a simple general equilibrium model, we can separate productivity and quality. Specifically, we can infer relative quality changes using time series of relative prices and budget shares. In addition, with a Vector Auto Regressive (VAR) model, we can analyze the dynamic relationship between productivity and quality. The empirical results based on US data show that quality innovation plays an important role in variations of the services-goods relative price, and productivity shock alone cannot fully explain the behavior of this relative price. This implies that models with only productivity shocks may generate misleading results. Stockman & Tesar (1995) reported that the addition of a taste shock between tradeables and non-tradeables helps better explain some international stylized facts which are hard to arrive at with productivity shock alone.

There is a new and growing literature on inferring quality from price information. Klenow (2003) provided a detailed critical review on the efforts by different statistical agencies in separating quality improvements from price changes, and gave some practical suggestions. Bils & Klenow (2004) decomposed inflation in unit prices of 66 US durable consumer goods into quality and pure-price effects. Hummels & Klenow (2005) looked at many countries' detailed exports data and found that richer countries charge higher prices which result from better quality. Hallak (2006) retrieved quality from export unit prices at the sectoral level and confirmed the theoretical prediction that rich countries buy relatively more from countries of high quality goods. The major weakness of this literature is that quality is retrieved either with a focus on prices alone or with inadequate specifications of sectoral production.

In this empirical quality literature, the closest work to the current study is Hallak & Schott (2005). They used relative prices and sectoral trade balances to decompose export unit value into quality and non-quality components. Their argument is: given a common international price for some sector, a country has a positive trade balance for that sector if its quality is higher than that of the trading partner. This argument does not always hold. For example, in a simple world where quality levels are the same and the law of one price holds, we can see non-zero sectoral trade balances: subject to disproportional quantities of sectoral endowments, countries benefit from net selling some product and net purchasing another. Moreover, their argument is hard to be extended to the aggregate level: overall trade balances partly reflect intertemporal consumption smoothing, which is not related to quality.

Our study has an aspect similar to that of the huge literature on demand empirics: looking at implications of utility maximization. However, the objectives of our study and those of the literature on demand empirics are different. The empirical demand literature has two major lines: (i) parametric approach in different flexible forms, e.g. the path-breaking paper by Diewert (1971) and a good empirical comparison by Fisher et al. (2001); and (ii) nonparametric approach with the generalized axiom of revealed preferences, e.g. Varian (1982, 1983). Both of these lines are concerned with the consistency between preference axioms and aggregate data. Our focus is on how aggregate quality is changing over time. Under the hypothesis that quality does change, the tests in parametric and nonparametric approaches have some problems. First, in different flexible forms of utility or indirect utility, consumption quantities are the only objects that evolve over time and all parameters are fixed. If quality and hence marginal utility are evolving, the parameters in those specifications should change, i.e. each period has some preference structure which is consistent with data of that period only. Consequently, with intensity in parameters, different flexible forms are hard to be properly implemented with aggregate data to capture quality innovation. Second, if quality varies and therefore the set of commodities evolve over time, we cannot apply the test of generalized axiom of revealed preferences when preferences may be quite different between periods. For these reasons, we rely on the parsimony in parameters to track quality changes and choose the constant elasticity of substitution (CES) utility function.

We focus on relative productivity and quality at the aggregate level and have some contributions to the literature. First, separation of productivity shock and quality innovation is based on relative prices and budget shares. This means that the retrieval of quality innovation fully takes preference and technology into account. Second, measures for goodness of fit, which tell how much productivity shock and quality innovation explain relative price and budget share, are developed. Based on these measures, we also know how important the measurement errors are in a specific economic context. Third, via an application, we know the evolution of US services-goods relative quality from 1946-2006. Fourth, by imposing a VAR structure on relative productivity shock and quality innovation, we have some insights on their individual and joint properties, which will serve as moments for further studies.

The rest of the chapter is structured as follows. Section 3.2 lays out the basic environment and focuses on productivity and quality information possibly borne by relative price variations. Section 3.3 extends the basic model by using both relative price and budget share to deal with measurement problems. The main result is an inference procedure for retrieving the relative quality index. Section 3.4 applies the methods developed earlier to US services-goods data. From this empirical analysis, we learn about the evolution and importance of quality innovation in the US context. Finally, we close the study with some remarks.

3.2 An Endowment Economy

3.2.1 The Basic Model

We have an economy populated by a unit measure of identical agents. In each period, the agents are endowed with commodities a and b and they can freely trade those endowments to satisfy their need. A typical agent $i \in [0,1]$ solves the following static CES utility maximization problem at time t

$$\max_{\{a_{it}, b_{it}\}} \left\{ (\alpha_t a_{it})^\theta + (\beta_t b_{it})^\theta \right\}^{1/\theta} \tag{3.1}$$

subject to the budget constraint

$$a_{it} + p_t b_{it} = e_{ait} + p_t e_{bit}, \tag{3.2}$$

where (a_{it}, b_{it}) are consumption quantities; (α_t, β_t) are positive quality indices of commodities a and b, respectively; (e_{ait}, e_{bit}) are endowment quantities; p_t is the relative price which denotes the amount of commodity a needed to trade for a unit of commodity b; and $\theta = 1 - 1/\sigma$, where σ is the constant elasticity of substitution (absolute value). In the literature, θ is called the *substitution parameter*. As the elasticity of substitution σ belongs to $[0, \infty)$, the substitution parameter θ lives in $(-\infty, 1]$. Essentially, we have a CES utility function in which effective consumption quantity is a product of quantity and quality. In this paper, changes in (α_t, β_t) are interpreted as quality innovation rather than taste shock. It is hard to interpret taste shock as a synchronized event happening to all agents, especially with a time length unit of one year or more. However, quality innovation can come from competition and imitation in production. With the restricted space of $\{\alpha_t, \beta_t, \theta\}$, marginal utilities are positive and decreasing. In addition, the utility function satisfies the Inada condition.

The CES specification in (3.1) covers a broad range of substitutability. Arrow et al. (1961) showed that: (i) CES is fixed-proportion Leontief ($\sigma = 0$) for $\theta = -\infty$; (ii) CES is inelastic ($0 < \sigma < 1$) for $\theta \in (-\infty, 0)$; (iii) CES becomes Cobb-Douglas ($\sigma = 1$) for $\theta = 0$; (iv) CES is elastic ($1 < \sigma < \infty$) for $\theta \in (0, 1)$; and CES has straight-line indifference curves ($\sigma = \infty$) for $\theta = 1$. In addition, the desired budget share for, without loss of generality, commodity a is a positive function of the coefficient α_t^θ.

On the technology side, total endowment quantities in any period t are

$$E_{at} = A_t \tag{3.3}$$

$$E_{bt} = B_t, \tag{3.4}$$

where the quantity ratio B_t/A_t follows some stochastic process. As the agents equally share the endowments, $e_{ait} = A_t$ and $e_{bit} = B_t$ for every i and t. Besides quantity, the quality ratio β_t/α_t also evolves stochastically. In this study, A_t and B_t are mentioned as productivity.

Let $\omega_t = \{A_t, B_t, \alpha_t, \beta_t\}$ be the information set. The timing in period t is: (i) at the beginning of the period, quality indices and quantity shocks in ω_t are fully observed by all agents; (ii) based upon this information set, the agents figure out their consumption plans; (iii) and then they trade with competitive terms in the markets.

Definition 3.2.1. *A competitive equilibrium consists of the quantity and price functions* $\{a_{it}(\omega_t), b_{it}(\omega_t), p_t(\omega_t)\}_{i\in[0,1],t\geq 0}$ *which satisfy the following conditions in any period* t*:*

(i) Given some information set ω_t *and price* p_t*,* $\forall i$*,* $\{a_{it}, b_{it}\}$ *maximize agent* i*'s utility in (3.1) subject to the budget constraint in (3.2);*

(ii) Given the information set ω_t *and the consumption plans in (i), price* p_t *clears the markets:*

$$\int_0^1 a_{it} di = A_t \tag{3.5}$$

$$\int_0^1 b_{it} di = B_t. \tag{3.6}$$

As all agents are identical, $a_{it} = A_t$ and $b_{it} = B_t\ \forall i$. After some manipulations (Appendix B.1), we derive the equilibrium relative price as

$$p_t = \left(\frac{\beta_t}{\alpha_t}\right)^{\theta} \left(\frac{B_t}{A_t}\right)^{\theta-1} \tag{3.7}$$

with the first-order derivatives

$$\frac{\partial p_t}{\partial \left(B_t/A_t\right)} = \left(\theta - 1\right) \frac{p_t}{B_t/A_t}, \tag{3.8}$$

$$\frac{\partial p_t}{\partial \left(\beta_t/\alpha_t\right)} = \theta \frac{p_t}{\beta_t/\alpha_t}. \tag{3.9}$$

There are some terminological notes. First, sector 2 has a *favorable productivity shock* if the ratio B_t/A_t is higher than that in the previous period. Second, sector 2 has a *favorable quality innovation* if the ratio β_t/α_t becomes higher. These notes also apply to sector 1 with respect to the ratios A_t/B_t and α_t/β_t. Third, *relative price* of a sector tells how many units of the other commodity needed to trade for one unit of this sector's commodity. We have the following results:

Proposition 3.2.2. *Ceteris paribus,*

(i) when a sector has a favorable productivity shock, its relative price depreciates if $\theta < 1$, and stays the same if $\theta = 1$;

(ii) when a sector has a favorable quality innovation, its relative price depreciates if $\theta < 0$, remains unchanged if $\theta = 0$, and appreciates if $\theta \in (0, 1]$.

Proof. These results can be directly inferred from (3.8) and (3.9). ∎

Table 3.1 summarizes the results in Proposition 3.2.2. We clearly see the qualitative effects of two sources of variations on the equilibrium relative price. It is interesting to note that when $0 < \theta < 1$, a favorable productivity

Table 3.1: Partial effects of different variations on relative price

	$\theta < 0$	$\theta = 0$	$0 < \theta < 1$	$\theta = 1$
favorable productivity shock	$-$	$-$	$-$	0
favorable quality innovation	$-$	0	$+$	$+$

shock and a favorable quality innovation have opposite effects on relative price. The following discussions will provide more intuition behind these results.

Partial Effects of Productivity Shock. When $\theta < 1$ or the utility function is strictly concave, an increase in the relative endowment of either commodity will eventually push down the marginal utility of that commodity relative to the other's, and hence the relative price decreases. When $\theta = 1$ or the commodities are linearly substitutable, marginal utility is constant given some quality indices, and relative price is not affected by productivity shock. Generally, relative productivity and relative price have a negative correlation.

Partial Effects of Quality Innovation. When $\theta < 0$, an increase in relative quality of either commodity will push down the relative price of that commodity, given some endowments. Interestingly, this result seems counter-intuitive at the face value. To see why the result is true, note that this is the case where products are hard to be substituted, which means the utility function is quite concave with respect to quantity and quality (Figure B.1). Thus, an increase in quality may shift up the utility function faster at low quantity than at high quantity, reducing marginal utility at each quantity level. The opposite effects apply for $\theta > 0$ (Figure B.2). That is an increase in quality may shift up the utility function faster at high quantity than at low quantity, increasing marginal utility at each quantity level. When $\theta = 0$, we have a Cobb-Douglas utility function where quality innovation does not affect relative marginal utility.

In addition to the equilibrium relative price p_t, we have the equilibrium budget share for commodity a (S_{at}) as

$$S_{at} = \frac{1}{1 + p_t \frac{B_t}{A_t}}, \text{ or} \tag{3.10}$$

$$S_{at} = \frac{1}{1 + \left(\frac{\beta_t}{\alpha_t}\right)^{\theta} \left(\frac{B_t}{A_t}\right)^{\theta}}. \tag{3.11}$$

It is noted for expositional simplicity, we use S_{at} rather than S_{bt}. From (3.7) and (3.11) we see that variations in productivity shock and quality innovation are manifested by both relative price and budget share. This result is coined *double manifestation*. Thus, if productivity shock, relative price, and budget share can be perfectly observed, there are two alternative ways to infer quality innovation, i.e. either with relative price or budget share. For example, given relative quantity and price data, we can estimate the substitution parameter with a regression, possibly with some instruments, as follows

$$\ln p_t = \lambda_0 + (\theta - 1) \ln(B/A)_t + \varepsilon_t. \tag{3.12}$$

and calculate the quality index as

$$(\beta/\alpha)_t = \left[p_t / \left(B/A\right)_t^{\theta-1}\right]^{1/\theta}. \tag{3.13}$$

If we have a world of two countries each endowed with one of the commodities a and b, and free trade takes place, the equilibrium terms of trade will also have the form in (3.7). Thus, even though the model is explicitly about a closed economy, its essentials can be extended to international contexts. However, those potential extensions will need many additional considerations, which are beyond the scope of this study.

3.2.2 VAR and a Dynamic Relationship

We may see different correlation patterns between productivity shock and quality innovation, as long as we already know both of the series, hereafter,

$(B/A)_t$ and $(\beta/\alpha)_t$. We choose a simple VAR model to analyze the dynamic relationship between them. It is noted that the VAR model is not used to separate quality innovation. It is used to generate some moments of interest.

We are developing a simple procedure to learn about variance, persistence, causation, and correlation of productivity shock and quality innovation, which are assumed to follow a lag-1 VAR model. The quantity process is characterized by mean μ_p and standard deviation (STD) σ_p. The quality process has mean μ_q and STD σ_q. Quantity and quality have a correlation coefficient of φ and a corresponding covariance of $\sigma_{pq} = (\sigma_p \sigma_q)\varphi$. We construct two new random variables as deviation from mean

$$P_t = (B/A)_t - \mu_p \tag{3.14}$$

$$Q_t = (\beta/\alpha)_t - \mu_q. \tag{3.15}$$

By construction, $var(P_t) = \sigma_p^2$, $mean(P_t) = 0$, $var(Q_t) = \sigma_q^2$, $mean(Q_t)$ $= 0$, and $covar(P_t, Q_t) = \sigma_{pq}$. The VAR model for (P_t, Q_t) is specified as

$$\begin{bmatrix} P_t \\ Q_t \end{bmatrix} = \begin{bmatrix} \lambda_{pp} & \lambda_{qp} \\ \lambda_{pq} & \lambda_{qq} \end{bmatrix} \begin{bmatrix} P_{t-1} \\ Q_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{pt} \\ \varepsilon_{qt} \end{bmatrix}, \tag{3.16}$$

where

$$\begin{bmatrix} \varepsilon_{pt} \\ \varepsilon_{qt} \end{bmatrix} \overset{i.i.d}{\sim} N(\mathbf{0}, \Sigma) \text{ and } \Sigma = \begin{bmatrix} \gamma_p^2 & \gamma_{pq} \\ \gamma_{pq} & \gamma_q^2 \end{bmatrix}. \tag{3.17}$$

For the sake of simulations, we need to specify Σ based upon characteristics of (P_t, Q_t). We have the following relationship (Appendix B.2)

$$\begin{bmatrix} \gamma_p^2 \\ \gamma_q^2 \end{bmatrix} = \begin{bmatrix} 1-\lambda_{pp}^2 & -\lambda_{qp}^2 \\ -\lambda_{pq}^2 & 1-\lambda_{qq}^2 \end{bmatrix} \begin{bmatrix} \sigma_p^2 \\ \sigma_q^2 \end{bmatrix} - \begin{bmatrix} 2\lambda_{pp}\lambda_{qp}\sigma_{pq} \\ 2\lambda_{pq}\lambda_{qq}\sigma_{pq} \end{bmatrix}, \tag{3.18}$$

$$\gamma_{pq} = \left[1 - (\lambda_{pp}\lambda_{qq} + \lambda_{pq}\lambda_{qp})\right]\sigma_{pq} - \left(\lambda_{pp}\lambda_{pq}\sigma_p^2 + \lambda_{qp}\lambda_{qq}\sigma_q^2\right). \tag{3.19}$$

Equations (3.18) and (3.19) convert the original parameters into error parameters of the VAR process. Besides guaranteeing $\left\{\sigma_p^2, \sigma_q^2\right\}$ to be finite, the VAR coefficients are further restricted so that the computed variances in (3.16) be non-negative and $|\gamma_{pq}|/(\gamma_p\gamma_q) \leq 1$.

By looking at the structure specified in (3.16) and (3.17), we know which of the two processes are more volatile and more persistent. In addition, we know their correlation and causation relationships. First, it is noted that the VAR structure nests the independence case. Thus we can test to see if the two processes are independent or not. If correlation of the errors and off-diagonal coefficients in the VAR model are statistically small, productivity shock and quality innovation can be considered independent. The test of correlation between the VAR errors is not straightforward because we do not know the variances of the estimated variance-covariance matrix $\widehat{\Sigma}$. It is noted that estimation of the VAR structure brings about unbiased estimates of the VAR coefficients and Σ. Thus we can employ the unbiasedness to simulate many samples and come up with different estimates of Σ and calculate variances of $\widehat{\Sigma}$. Second, the test of causation between quantity and quality can be simply implemented with t-tests on the off-diagonal VAR coefficients.

3.3 From Ideology to Data

In the basic model, it is straightforward to calculate the quality index. Potentially, there is a mismatch to some extent between the basic model, which is an ideology, and data for two major reasons. First, actual economic contexts do not satisfy all the underlying assumptions of the basic model. Second, observations of relative price, productivity shock, and budget shares are not perfect. Productivity shock and relative price may be imperfectly observed for many reasons, e.g. under-reporting and aggregating over heterogenous and evolving types of commodities. In this section, we discuss what economic contexts the model can be applied to, and consider several ways to deal with imperfect observability and infer quality innovation.

3.3.1 A Valid Data Set

For an empirical implementation of the basic model, an actual economic context or a data set should possess three critical properties as follows.

Relative Completion. First, the sample should reflect a relatively closed system. To put it differently, variations in relative quantity and price should not be largely influenced by supply and demand outside the economy. If relative completion is violated, relative prices do not bear reliable information on the system's fundamentals, i.e. productivity shock and quality innovation.

Full Equilibrium. Second, variations in nominal prices should fully reflect changes in productivity shock and quality innovation. In equation (3.7), we see that, relative price, which will be constructed based upon nominal prices, has to adjust to clear commodities markets in equilibrium. If nominal prices are not free to move, relative price does not provide good information on variations deep in the economic system. This also implies that we should not look at high frequency data which potentially have short-run deviations from the fundamentals due to many reasons, e.g. nominal rigidities, unbalanced monetary effects, and speculations. Besides the price-adjustment concern, frequency of data should be low enough for full response of commodity supply and delivery. In other words, data should reflect a system in equilibrium rather than on-going adjustment.

CES Compatibility. Third, the estimated substitution parameter $\widehat{\theta}$ should lie in the interval $(-\infty, 1]$ to be consistent with the CES specification. Recall that $\widehat{\theta}$ is the estimator in a regression, possibly an IV regression, with relative price as the dependent and relative quantity as the independent. If relative quantity and relative price move in opposite directions, $\widehat{\theta}$ is highly likely negative and readily valid. If relative quantity and relative price have positive correlation, the latter should not be too volatile in comparison with the former so that $\widehat{\theta}$ is smaller than unity. In other words, the CES specification is not compatible with too volatile relative price which is positively correlated with productivity shock.

3.3.2 Matching only with Relative Price

There are several ways to utilize the double manifestation result. One is to retrieve quality innovation only from relative price according to (3.13), and check how well the model budget share matches with its data counterpart. When applying the basic model to real economic contexts, the inferred quality series may not be totally consistent with the observed budget share. Here are some possible reasons for this potential inconsistency. First, in empirical analyses, the normalized and indexed world only maintains the true growth rates of relative quantity and relative price rather than their true levels. This implies that the computed budget share as defined in (3.11) does not necessarily match with data counterparts. All we can check is the correlation between them. Second, as mentioned earlier, productivity shock and quality innovation are not perfectly observed. Third, the basic model does not have investment. In reality, this is not the case. Among the three problems mentioned, we will tackle the first and second in the next sections.

Alternatively, we can use budget share data to infer quality and check the result with relative price data. For this, the starting point is (3.11).

3.3.3 Double Manifestation and Rescaling

In reality, we often observe productivity shock and relative price as indices. The double manifestation will help us rescale these indices to make model budget share close to its data counterpart. Explicitly, let u and s be the correct rescaling constants. We observe index $(B/A)_t$ for productivity shock and the true productivity shock is $(B/A)_t\, u$. By the same token, we observe index p_t and rescale it to the true level $p_t s$. Expressions (3.7) and (3.11) are rewritten as

$$p_t s = \left(\frac{\beta}{\alpha}\right)_t^{\theta} \left[\left(\frac{B}{A}\right)_t u\right]^{\theta-1}$$

$$S_{at} = \frac{1}{1 + (p_t s)\left[\left(\frac{B}{A}\right)_t u\right]},$$

or

$$p_t u s = \left[\left(\frac{\beta}{\alpha}\right)_t u\right]^{\theta} \left(\frac{B}{A}\right)_t^{\theta-1} \tag{3.20}$$

$$S_{at} = \frac{1}{1 + (p_t u s)\left(\frac{B}{A}\right)_t}. \tag{3.21}$$

In equation (3.21), it can be seen that the product us can be estimated. However, u and s cannot be individually identified. That also means that we can only infer quality innovation correct up to some unknown scale u, which is used for rescaling productivity shock.

Specifically, the product us is chosen to minimize the squared differences between the model and data budget shares for commodity a as

$$\widehat{us} = \arg\min_x \sum_{t=1}^{T} \left(\left[\frac{1}{S_{at}} - 1\right] - \left[p_t \left(\frac{A}{B}\right)_t\right] x\right)^2. \tag{3.22}$$

After some simple manipulations, we have the optimal rescaling constant

$$\widehat{us} = \frac{\sum_{t=1}^{T} \left[\frac{1}{S_{at}} - 1\right] \left[p_t \left(\frac{A}{B}\right)_t\right]}{\sum_{t=1}^{T} \left[p_t \left(\frac{A}{B}\right)_t\right]^2}. \tag{3.23}$$

Next, we rescale p_t with us and estimate θ with an IV estimation. Finally, quality innovation is calculated according to (3.24).

$$\left(\frac{\beta}{\alpha}\right)_t u = \left[\frac{p_t \widehat{us}}{(B/A)_t^{\theta-1}}\right]^{1/\theta}. \tag{3.24}$$

Note that estimates for θ are the same for original and rescaled data. Thus, with this rescaling scheme, double manifestation is satisfied by the model to some extent. If we have good level data for relative quantity or relative price, u can be calculated and relative quality will be rescaled to the true level.

3.3.4 Allowing for Measurement Errors

In the previous section we see that rescaling helps match with the data budget share for a to some extent. In this section, we still use this rescaling scheme and add the unknown factors (u_{1t}, u_{2t}) as in (3.25) and (3.26). Expression (3.25) comes from (3.20), and equation (3.26) is derived from (3.21). For expositional simplicity, we look at the modified budget share rather than the original one. The motivation for these errors is that there are measurement errors in productivity shock, relative price, and budget share. In addition, these are perfectly observed by the agents and not by econometricians. With a multiplicative error structure, imperfect observability is embedded in (u_{1t}, u_{2t}).

$$p_t us = \left[\left(\frac{\beta}{\alpha}\right)_t u\right]^{\theta} \left(\frac{B}{A}\right)_t^{\theta-1} u_{1t} \tag{3.25}$$

$$\frac{1}{S_{at}} - 1 = \left[\left(\frac{\beta}{\alpha}\right)_t u\right]^{\theta} \left(\frac{B}{A}\right)_t^{\theta} u_{2t}. \tag{3.26}$$

Without multiplicative constants, we do not impose that $E(u_{1t}) = E(u_{2t}) = 1$. However, (u_{1t}, u_{2t}) is assumed to have a finite variance-covariance matrix. It is noted that the scale us is a function of observables as in (3.23) and θ can be estimated by an IV estimation according to (3.12).

Given a static world where there are no intertemporal choices, we choose $[(\beta/\alpha)_t u]^{\theta}$ to minimize the objective function in (3.27) for any period t

$$\left[\left(\frac{\beta}{\alpha}\right)_t u\right]^{\theta}_{est} = \arg\min_x \left\{U_t' W U_t\right\} \tag{3.27}$$

where

$$U_t = \begin{bmatrix} \widetilde{u}_{1t} \\ \widetilde{u}_{2t} \end{bmatrix},\ \widetilde{u}_{1t} = \frac{1}{u_{1t}} - 1,\ \widetilde{u}_{2t} = \frac{1}{u_{2t}} - 1,$$

$$u_{1t} = \frac{1}{C_{1t} x},\ C_{1t} = \frac{(B/A)_t^{\theta-1}}{p_t us},$$

$$u_{2t} = \frac{1}{C_{2t} x},\ C_{2t} = \frac{(B/A)_t^{\theta}}{1/S_{at} - 1},$$

and

$$W = \Omega^{-1}$$
$$\Omega = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$$
$$\Omega^{-1} = \frac{1}{Det\left(\Omega\right)} \begin{bmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{bmatrix}$$
$$var\left(\widetilde{u}_{1t}\right) = \sigma_1^2,\ var\left(\widetilde{u}_{2t}\right) = \sigma_2^2,\ covar\left(\widetilde{u}_{1t}, \widetilde{u}_{2t}\right) = \sigma_{12},$$
$$Det\left(\Omega\right) = \sigma_1^2\sigma_2^2 - \sigma_{12}^2 = \sigma_1^2\sigma_2^2\left(1 - \frac{\sigma_{12}^2}{\sigma_1^2\sigma_2^2}\right) > 0.$$

It is straightforward to have the minimizer in (3.27) as (Appendix B.3)

$$\left[\left(\frac{\beta}{\alpha}\right)_t u\right]_{est}^{\theta} = \frac{\sigma_2^2 C_{1t} + \sigma_1^2 C_{2t} - \sigma_{12}\left(C_{1t} + C_{2t}\right)}{\sigma_2^2 C_{1t}^2 + \sigma_1^2 C_{2t}^2 - 2\sigma_{12} C_{1t} C_{2t}}, \tag{3.28}$$

and hence the estimator of relative quality index is

$$\left[\left(\frac{\beta}{\alpha}\right)_t u\right]_{est} = \left[\frac{\sigma_2^2 C_{1t} + \sigma_1^2 C_{2t} - \sigma_{12}\left(C_{1t} + C_{2t}\right)}{\sigma_2^2 C_{1t}^2 + \sigma_1^2 C_{2t}^2 - 2\sigma_{12} C_{1t} C_{2t}}\right]^{1/\theta}. \tag{3.29}$$

Let q_t be the true relative quality index in period t and let $\Phi\left(\widetilde{u}_{1t}, \widetilde{u}_{2t}; q_t\right)$ be the estimator in (3.29). The estimator of relative quality index has the following conditional expectation and variance

$$E\left[\Phi\left(\widetilde{u}_{1t}, \widetilde{u}_{2t}; q_t\right)|q_t\right] = q_t E\left\{\left[\frac{\Phi_{Ut}}{\Phi_{Lt}}\right]^{1/\theta}\right\} \tag{3.30}$$

$$var\left(\Phi\left(\widetilde{u}_{1t}, \widetilde{u}_{2t}; q_t\right)|q_t\right) = \Delta\Phi\left(\widetilde{\mu}_1, \widetilde{\mu}_2; q_t\right)' \Omega \Delta\Phi\left(\widetilde{\mu}_1, \widetilde{\mu}_2; q_t\right), \tag{3.31}$$

where $\widetilde{\mu}_1 = E\left(\widetilde{u}_{1t}\right)$, $\widetilde{\mu}_2 = E\left(\widetilde{u}_{2t}\right)$, and $\left(\Phi, \Phi_{Ut}, \Phi_{Lt}\right)$ are defined in (B.14) (Appendix B.3). The complex term $E\{[\Phi_{Ut}/\Phi_{Lt}]\}^{1/\theta}$ in (3.30) is called the *correction factor* whose sample counterpart is defined in (B.17). If the sample correction factor is significantly different from unit, the point estimator and variance in (3.29) and (3.31) should be adjusted accordingly.

There are two important notes. First, we do not know Ω at the start. For this reason, the implementation procedure has two steps. In the first step,

Ω_1 is the identity matrix. In the second step, Ω_2 is established based upon the estimated errors $\{\widehat{\widehat{u}}_{1t}, \widehat{\widehat{u}}_{2t}\}_{t=1}^{T}$ from the first step (Appendix B.3). In implementation, we can actually repeat the steps until the estimated Ω converges, given a small tolerance level. Second, in the inference procedure, we use two pieces of information to pin down relative quality. This may lead to overidentification. The overidentification test is carried out based on the standard J-statistic which is Chi-squared distributed with one degree of freedom.

The corresponding fitted relative price and budget share are

$$(p_t us)_{fit} = \left[\left(\frac{\beta}{\alpha}\right)_t u\right]_{est}^{\theta} \left(\frac{B}{A}\right)_t^{\theta-1} \tag{3.32}$$

$$(S_{at})_{fit} = \frac{1}{1+\left[\left(\frac{\beta}{\alpha}\right)_t u\right]_{est}^{\theta} \left(\frac{B}{A}\right)_t^{\theta}}. \tag{3.33}$$

As $(\widehat{u}_{1t}, \widehat{u}_{2t})$ capture the differences between model outcomes and data counterparts, we define two goodness-of-fit measures

$$\text{relative price} \quad : \quad R_{RPR} = 1 - \left(\frac{1}{T}\sum\nolimits_{t=1}^{T} \left[\widehat{u}_{1t} - 1\right]^2\right)^{1/2}, \tag{3.34}$$

$$\text{budget share for } a \quad : \quad R_{BSA} = 1 - \left(\frac{1}{T}\sum\nolimits_{t=1}^{T} \left[\widehat{u}_{2t} - 1\right]^2\right)^{1/2}. \tag{3.35}$$

By construction R_{RPR} and R_{BSA} generally live in $[0,1]$ and tell how much variation in relative price and budget share is explained by productivity shock and quality innovation. In addition, the quantitative role that measurement errors play in a specific context is captured by $(1 - R_{RPR})$ and $(1 - R_{BSA})$.

3.4 US Services vs. Goods in 1946-2006

In this section, we look at relative productivity shock and quality innovation between two US broad product groups: services and goods, respectively commodities b and a in the theoretical model. The annual data set, which is drawn from the National Income and Product Accounts (NIPA), covers the period 1946-2006 (Appendix B.4).

3.4.1 Data Description

The data set is valid for the basic model because it satisfies the three critical conditions. First, we can treat the US economy as being relatively closed. Net exports play a small part in total GDP, i.e. 3.2 percent in 1946, -5.8 percent in 2006, and -0.7 percent on average in 1946-2006 (Table B.1). Second, annual data is expected to allow full adjustments in most real activities and nominal prices. Third, we will see that the estimated substitution parameter $\widehat{\theta} \leq 1$, satisfying the CES specification.

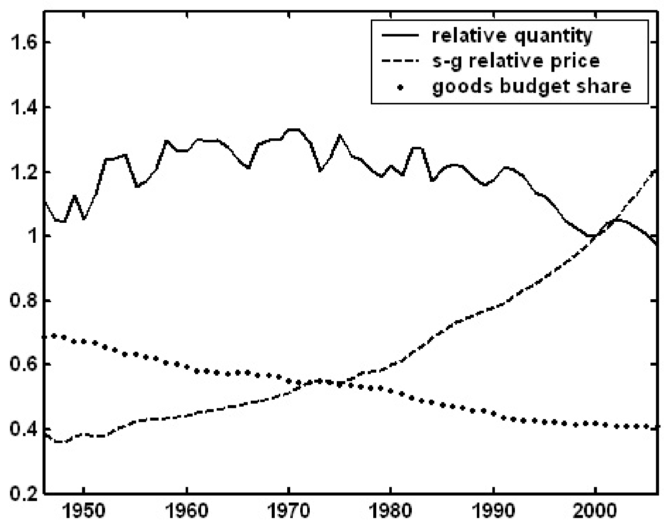
Table 3.2: Variables in US data set

Description	Definition
Goods quantity index	Q_G
Goods price index	P_G
Services quantity index	Q_S
Services price index	P_S
Budget share for goods*	BSG
US population index*	POP
Services-goods relative quantity*	$SGP = Q_S/Q_G$
Services-goods relative price*	$RPR = P_S/P_G$

Note: (*) unit root at 5%; see Appendix B.4 for more details.

Here are some important details on the construction and use of the variables (Table 3.2). First, quantity and price indices are constructed with a Fisher's formula, which uses weights from two adjacent years (Appendix B.4). In addition, quantity variables include final sales of domestic product and changes in inventories, and exclude imports. It is noted that we exclude structures in all considerations because they have service flows for an extended period of time, which is hard to be picked up by a static model (Appendix B.4). Second, the budget share for goods is calculated based on private consumption data, which does not include investment, and covers imported goods and services for consumption. It is noted that there is currently no reliable information to separate domestic and imported products in private consumption. As mentioned earlier, we can treat the US economy as relatively closed. Third, US population will be used as the instrumental variable in the estimation of θ. Relative quantity is expected to bear some information about total population. In the mean time, we do not expect a relationship between relative quality and population. Later in the implementation, we will check if total population is a valid instrument.

Figure 3.1: US data set 1946-2006, year 2000 = 1



Source: constructed from NIPA (BEA).

Time series of relative quantity, relative price, and budget share for goods in 1946-2006 are presented in Figure 3.1. It can be observed that budget share for goods is decreasing over time. In addition, while relative quantity of services is fluctuating, the relative price has an increasing trend. This latter observation suggests quality innovation may have some effects on the relationship between relative price and productivity shock.

3.4.2 Quality Information in Data

We have some further notes about the quality information and other noisy information possibly borne by price and budget share data.

First, the current statistical system measures a value index as the product of price and quantity indices, e.g. US BEA's method in (B.18). Let a be physical quantity associated with physical price p. Let αa be efficiency quantity associated with efficiency price $\widehat{p}$. We observe that the value index can be interpreted in different ways, i.e. $p.a = \widehat{p}.\alpha a$. Thus, if we deflate the value index by some price deflator, physical or efficiency, we will have the corresponding quantity index. Conceptually, our quality inference methods rely on the physical price p because it has the quality content. The question is what price data do we currently have, physical price or efficiency price? The answer is a mixture of the two which is closer to physical price. In other words, price data bear information about quality changes to a large extent. It is noted that, the extent to which prices reflect quality is not fixed. There is a gradual evolution from p to $\widehat{p}$ by the moves of different US statistical agencies, especially the Bureau of Labor Statistics (BLS) whose consumer and producer price indices are used by the others. Before 1998, there were quality adjustments to some products like motor vehicles and apparels by the BLS. The Boskin Commission of 1996 reported that price indices are biased upward for not adjusting quality changes. Since 1998, the BLS has used hedonic price regressions more extensively to adjust quality changes in prices. The extent to which prices are adjusted for quality changes is far from complete. Landefeld & Grimm (2000) estimate that, by 2000, 18 percent of US final expenditure is deflated by hedonic prices. Thus the price and quantity data used in the current research are not conceptually perfect as the physical price and quantity, especially after 1998. However, as price data still bear much quality information, the quality inference exercise is valid.

Second, price data do not differentiate between quality improvement and variety growth. Quality improvement means consumers have higher utility from the same quantities of some fixed products. Variety growth means changes in the number of varieties while quality for each variety is constant. Theoretically, variety growth can be equivalently represented by quality improvement, e.g. total utility $\int_0^\theta u(x)\,di$ can be replaced by single utility $\theta u(x)$. Consequently, though explicitly about quality improvement, the basic model can also capture the effects of variety growth if price data bear these effects. In fact, the current statistical practice tends to support this. To see why, we look at an example of two cars of the same model. If they have the same color, each can be sold for ten thousand dollars. If they have different colors which are appreciated by consumers, each can claim eleven thousand dollars. In the second case, though the total quantity is the same, the average price is higher. In practice, the two car variants are recorded in the same category and the average price should bear information on variety growth.

Third, with annual data, we conjecture that the ratio between services and goods prices is not much biased by unbalanced monetary effects. Investigating a large sample in the US consumption price data for 1995-1997, Bils & Klenow (2004) show that it takes a median period of less than six months for prices to change. In addition, the relative frequency of price changes in all goods and services are 26 percent. Specifically, the relative frequencies of price changes for durable goods, nondurable goods, and services are respectively 30, 30, and 21 percent. Even though, the degree of nominal rigidity is not the same for all products, the probability that some price will change after one year is very large. That is, our data frequency is low enough for services and goods prices to bear equivalent monetary effects, and the relative price and budget share mostly capture relative productivity shock and quality innovation.

Fourth, we do not explicitly control production cost. However, production cost is linked to productivity and hence can be summarized by productivity shock. Thus, the basic model already somehow separates the cost effects on relative price and budget share.

Fifth, we currently do not have information on sales tax to refine price data. However, the tax information remaining in price data may be relatively harmless for several reasons: (i) we are interested in the ratio of two aggregate prices rather than individual price indices; (ii) at the aggregate level, the relative tax rates should be stable for two adjacent years; and moreover (iii) each link, i.e. year-to-year, in the Fisher price index series is not affected by a link far away from that. In other words, effective tax rates do not change much between two adjacent years, and the time series of services-goods relative price should bear noisy tax information only to a small extent relative to productivity and quality effects.

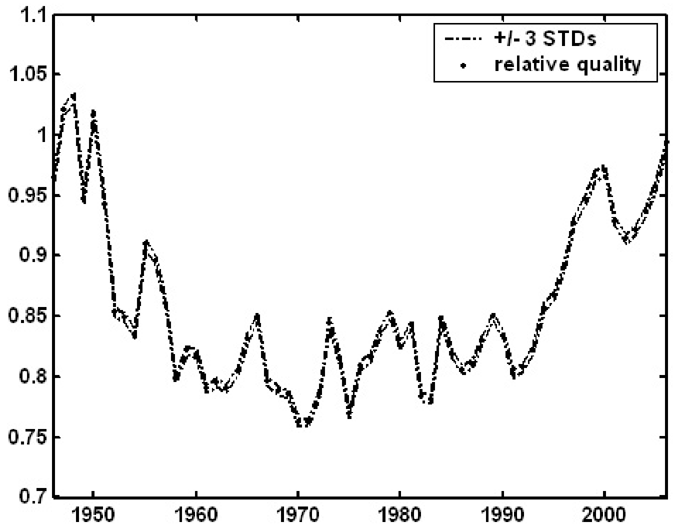
3.4.3 Services Relative Quality and Parameters

The implementation has three steps: (i) estimating the substitution parameter θ; (ii) inferring the quality index; and (iii) analyzing the dynamic relationship between productivity shock and quality innovation.

There are some notes about validity of the estimates. First, to check the validity of the instrument estimation following (3.12), we look at the correlation between relative quality and population growth. The correlation is weak at 4 percent. Meanwhile, productivity shock and population index time series are correlated at -43 percent. That means the estimate of θ is reliable. Second, the sample correction factor is found to be unit, i.e. the estimator is unbiased. Third, the J-test rejects the hypothesis of overidentification.

The point estimate for θ is -10, which means a substitution elasticity σ of 0.09 (simple OLS estimate for θ is -2.2, for σ is 0.3). That is goods and services are generally hard to substitute each other. Next, we calculate

Figure 3.2: Services-goods relative quality 1946-2006

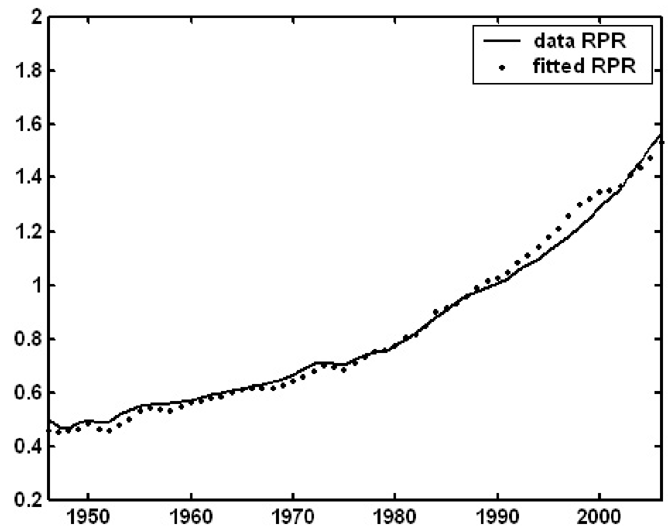


Note: estimation is based on (3.29) and (3.31).

the quality innovation time series with three methods: (i) matching only with relative price; (ii) rescaling relative price; (iii) and allowing for measurement errors as discussed in Section 3.3. The series generated by three methods have very high correlation. The striking result is that relative quality time series following method 2 and method 3 are very close. That means measurement errors play a small role in this specific case. The point estimate and $\pm 3STD$ band for services-goods quality index according to method 3 are presented in Figure 3.2. It can be seen that services relative quality was decreasing until early 1970s when it started increasing.

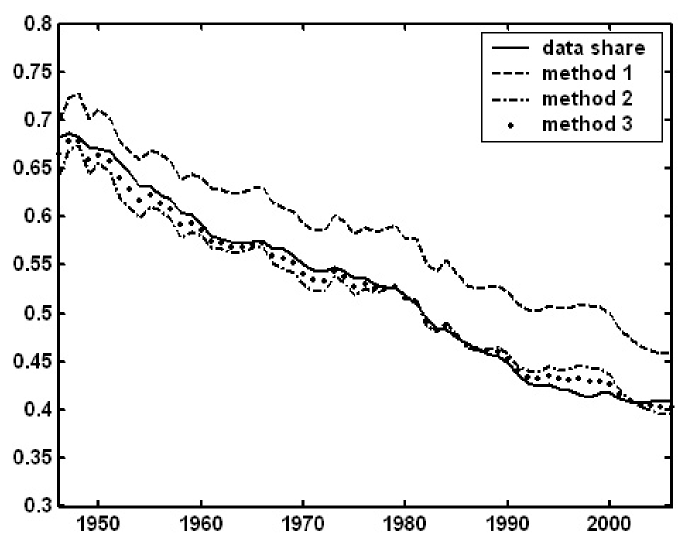
In overall, observed productivity shock and inferred quality innovation help largely explain variations in services-goods relative price and budget share for goods. In fact the measures for goodness of fit are very high, i.e. $R_{RPR} = 0.96$ and $R_{BSA} = 0.97$. Figures 3.3 and 3.4 (below) show that the actual and fitted variables are very close to each other.

Figure 3.3: Actual and fitted relative prices 1946-2006



Note: fitted RPR in (3.32).

Figure 3.4: Actual and fitted budget shares for goods 1946-2006



Note: method 1 in (3.13); method 2 in (3.21); method 3 in (3.33).

Table 3.3: US services-goods: estimation results

Description	Definition	Estimate	STD
	CES specification		
substitution parameter	θ	-10.087	1.238
elasticity of substitution	$\sigma = \frac{1}{1-\theta}$	0.090	
	quantity and quality series		
productivity shock: mean & STD	μ_p & σ_p	1.183	0.098
productivity shock: variation	σ_p/μ_p	0.083	
quality innovation: mean & STD	μ_q & σ_q	0.859	0.071
quality innovation: variation	σ_q/μ_q	0.082	
correlation coefficient	φ	-0.917	
covariance	$\sigma_{pq} = (\sigma_p\sigma_q)\varphi$	-0.006	
	VAR coefficients		
quantity on quantity	λ_{pp}	1.208	0.145
quality on quantity	λ_{qp}	0.425	0.200
quantity on quality	λ_{pq}	-0.182	0.115
quality on quality	λ_{qq}	0.650	0.158
	Σ specification		
variance of quantity error	γ_p^2	0.0020	$1.5e-4$
variance of quality error	γ_q^2	0.0013	$1.3e-6$
error covariance	γ_{pq}	-0.0016	$1.2e-5$
error correlation	$\frac{\gamma_{pq}}{\gamma_p\gamma_q}$	-0.9884	
	Correlation with relative price		
quantity series		-0.627	
quality series		0.281	
	Correlation with instrument–population		
quantity series		-0.430	
quality series		0.044	

Given both productivity shock and quality innovation time series, we now analyze them by the VAR model discussed earlier. Estimates of the VAR structure are presented in Table 3.3. From the results, we have several observations as follows. First, quality innovation is as volatile as productivity shock ($\sigma_q/\mu_q \approx \sigma_p/\mu_p$). Second, productivity shock is more persistent than quality innovation ($\lambda_{pp} > \lambda_{qq}$). Even though productivity shock is not stationary, we still use it level time series to generate the moments of interest. Third, quality innovation has positive effect on productivity shock while the latter has relatively small negative impact on the former (VAR coefficients). Fourth, productivity shock and quality innovation have negative correlation, which partly comes from large negative correlation between two technical seeds, i.e. error correlation is at -99 percent. This strong result suggests that there is an endogenous trade-off between productivity shock and quality innovation. Fifth, quantity error is more volatile than quality error. Sixth, relative price is correlated with productivity shock at -0.63 and with quality innovation at 0.28 percent. As negatively correlated, productivity shock and quality innovation weaken each other, leading to a less volatile relative price.

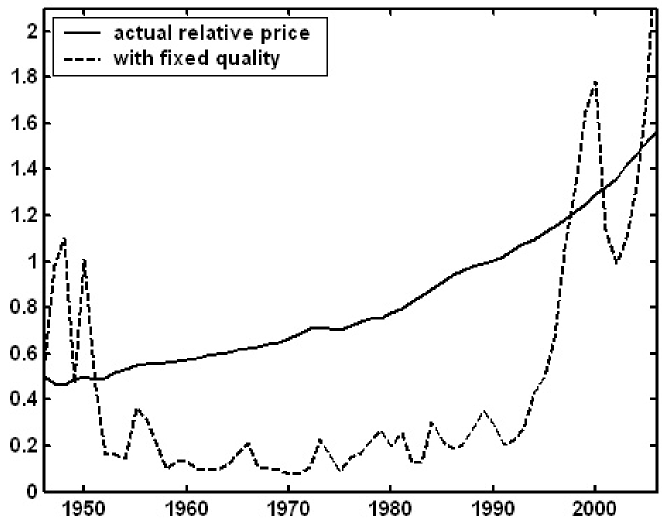
3.4.4 With and without Quality Innovation

To clearly see the role of quality innovation, we carry out two counterfactual analyses, one is on relative price and the other on relative quantity.

First is a counterfactual analysis on the relative price, in which quality index is kept constant and productivity shock alone drives the relative price. Figure 3.5 shows that productivity shock alone can produce the upward sloping in relative price to some extent. However productivity shock poorly projects the smoothness in relative price. This counterfactual result suggests that if we ignore quality innovation and try to reproduce some relative price, the result can be an estimated quantity series with different properties than reality.

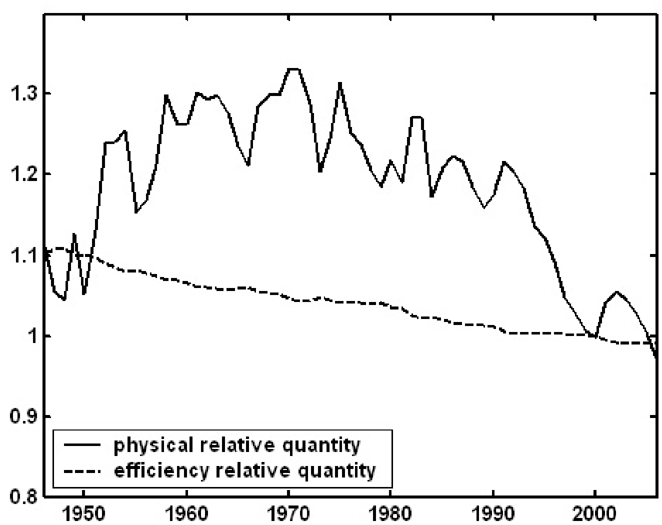
Second is a contrast between physical and efficiency quantities. Figure 3.6 shows that while physical quantity has a lot of variations, efficiency quantity is much smoother and has a negative trend. Again, this result puts forth a warning on empirical studies: we need the consistency between the objects in consumption, production, and the data counterparts in terms of quality nature. Conditional on questions of interest, an inconsistency between model and data objects may lead to misleading results.

Figure 3.5: Actual and counterfactual relative prices 1946-2006



Note: quality index in (3.32) is kept constant at 0.944.

Figure 3.6: Physical and efficiency quantities 1946-2006



Note: efficiency quantity equals physical quantity times quality.

3.5 Conclusion

The current study develops a model which accounts for variations in both relative quantity and quality between sectors, and potentially between countries. In the model, relative productivity shock and quality innovation are manifested in both relative price and budget share, i.e. double manifestation. In addition, partial effects of productivity shock and quality innovation on relative price and budget share depend on the substitution parameter. The double manifestation result helps separate the unobserved relative quality innovation. Given time series of productivity shock and quality innovation, we can investigate their individual and joint characteristics, i.e. variance, persistency, causation, and correlation.

We then apply the separation method to the services and goods sectors of the US from 1946-2006. The result shows that observed productivity shock and inferred quality innovation explain variations in the relative price and budget share very well. In addition, productivity shock alone fails to explain the smoothness in services-goods relative price. In this specific case, productivity shock and quality innovation are negatively correlated. Essentially, this is a specific case which supports the quality innovation hypothesis, i.e. aggregate quality does change over time.

The theoretical and empirical results put forth a warning that growth and business-cycle models should not ignore quality innovation at the start. Specifically, the missing of quality innovation may be relatively harmless in a certain set of growth and business-cycle models. However, by not explicitly modeling quality innovation, models with an emphasis on relative prices may generate misleading results. In a certain context, we need to evaluate the relative importance of quality innovation before simplifying the working model.

Klenow (2003) suggests that "...the BLS begin tabulating the difference between inflation in average unit prices and inflation in the CPI. This gap reflects BLS quality adjustments..." Our study supports this suggestion. More specifically, we should keep both price measures, one for physical price (p), and the other for efficiency price ($\widehat{p} = p/\alpha$). In statistical practice, we are now moving from measuring p towards measuring $\widehat{p}$. Efficiency price $\widehat{p}$ is directly useful for welfare calculations. However, if we do not keep physical price p, we loose the ability to separate quantity and quality. Moreover, if quantity and quality are lumped into one measure, we will have another model of the world. That means we may have some policy descriptions which do not deal with the differences between quantity production and quality innovation, possibly leading to non-optimal outcomes. The next chapter is a case where there are differences between quantity production and quality innovation.

Chapter 4

Productivity Driven Quality

4.1 Introduction

This study considers a fixed set of two products whose quality varies in response to changes in productivity (*productivity-driven quality* or *quality innovation* for short) in the context of competitive growth and business-cycle models. We then fit the growth model to US data to pin down some key parameters and generate a time path of relative quality (services versus goods).

At disaggregate levels, we can actually observe quality changes, e.g. better cars or new bank services coming out each year. However, at different aggregate levels, quality changes are not directly observable. Chapter 3 proposes a simple method which relies on relative prices and budget shares to infer relative quality changes. In that study, we treat relative quantity and quality as exogenous variations. Applying the inference method to the goods and services sectors of the US from 1946-2006, we see that quantity and quality of services (relative to goods) are negatively correlated, suggesting an endogenous link between productivity (which drives quantity) and quality. The current study makes progress by explicitly modeling this link.

We address two questions. First, how does productivity affect quality? Second, in the United States, how important is productivity-driven quality in total quality, where the latter varies due to all possible sources like randomly developed ideas for quality improvement as well as productivity? Answers to these questions are interesting because this is an area that we know little about. In addition, the answers are potentially important for policies related to factor markets, e.g. labor and capital income taxes. That is how policies will affect allocation of resources between quantity production and quality innovation.

In this model, labor is used for both quantity production and quality innovation. As quantity and quality can be substitutable in consumption, a productivity change induces a reallocation of labor between different activities, leading to quality variation. Productivity-driven quality is a reality rather than just a hypothesis. For example, teachers can impose limits on class sizes to improve teaching quality. Researchers may refrain from carrying out simultaneously too many projects to work more on each project. Thus, a productivity change may eventually vary output quality. There is a potential problem in fitting the model to price and budget data because money stock and velocity are not fixed. To overcome this difficulty, we divide the economy into two sectors and define objects in relative terms, i.e. services versus goods. This modeling choice helps enrich theoretical examinations and apply the model to available aggregate data.

Here are the major findings. First, via both analytical and computational approaches, we find that (relative) productivity's effect on (relative) quality depends on two key parameters, which govern how substitutable the products are (substitution parameter) and how easy it is to improve quality (innovation parameter). Specifically, productivity and quality have a negative correlation for low-range substitutability and a positive correlation for medium-range substitutability, where the upper bound of the medium range negatively depends on the innovation parameter. Second, we extend the model so that aggregate quality plays a role in quantity production to see if the standard growth accounting method remains valid. The results show no big differences between the baseline and extended models. The reason is that individual quality indices move in opposite directions, keeping aggregate quality stable. This implies that the baseline model is rich enough for empirical application.

Third, when applying the baseline model to the US services-goods economy from 1970-2006, we see that productivity-driven quality can play as much as one half in total quality. For comparisons, we apply the method in Chapter 3 to infer total quality. The significance of productivity-driven quality calls for more policy attentions at this type of quality responses. For example, it may be easier and cheaper to raise social welfare by shifting some resources from quantity production to quality innovation by means of taxation in an existing environment with sub-optimal allocations. Fourth, via moment matching, we pin down the key parameters for the US. The parameter estimates imply a negative correlation between relative productivity and quality of services.

Essentially, quality innovation serves as a propagation mechanism of productivity changes in growth and real business-cycle (RBC) models. In the vast literature on growth and real business cycles, productivity changes are used to explain variations in quantity and variety, e.g. Kydland & Prescott (1982) and Chatterjee & Cooper (1993), respectively. As far as we know, our model is among the first to explore the third branch which examines how productivity affects quality.

There are two lines of literature that embrace two dimensions of quality changes. One is the "quality ladder" literature in which varieties are fixed while quality of products is evolving with a constant step in any period. The other is the "variety growth" literature in which quality of a single product is fixed and the number of products is changing. Two good examples of these lines are Grossman & Helpman (1991) and Aghion & Howitt (1992), respectively. In addition, there are papers that attempt to combine these two literatures. In the current study, we fix the number of products and allow quality to be endogenously determined. We deviate from the "quality ladder" literature in two aspects: (i) markets are perfectly rather than imperfectly competitive; and (ii) quality level and innovation steps are continuous in the sense of "quality escalator" (Chapter 2). We choose "quality escalator" over "quality ladder" and "variety growth" for several reasons. First, "quality escalator" allows for continuous quality ratios, richer sets of innovation decisions, and a better fit in empirical applications than "quality ladder". Second, "variety growth" can be equivalently represented by "quality escalator". For example, total utility $\int_0^\theta u(x)di$ can be replaced by single utility $\theta u(x)$. With the statistical practice, though not perfect, that separates quantity growth from price changes and condenses the large product space into a small discrete set, "quality escalator" is more favored than "variety growth".

It is also worthwhile discussing the difference between taste shock and quality change. Taste shock is a random change in valuation of the same products. Quality change is a variation in the nature of the product, either endogenous or exogenous, that alters agents' valuation. At the aggregate level and with low frequency data, it is hard to interpret taste shock as a synchronized event happening to all agents. Meanwhile, aggregate quality change can be automatically achieved via competition and imitation. We assume that a time length of one year is enough for sectoral quality synchronization.

The remaining of the study is structured as follows. Section 4.2 sets up the model. Section 4.3 characterizes the relationship between productivity and quality. Section 4.4 applies the model to the US economy. Finally, Section 4.5 concludes with some remarks.

4.2 An Economy of Two Sectors

In this section, we look at primitives of the environment, a competitive equilibrium notion, and an extension of the baseline model.

4.2.1 Baseline Preference and Technology

The economy has one representative consumer and two competitive commodity sectors. Sectors 1 and 2 produce perfectly divisible products a and b, respectively. The consumer is endowed with one unit of labor which is perfectly mobile between the sectors. The numeraire is good a at time zero, i.e. $p_{a0} = 1$. The representative agent has the life-time expected utility

$$E_0 \sum_{t=0}^{\infty} \lambda^t u\left(a_t, b_t; \alpha_t, \beta_t\right), \ 0 < \lambda < 1 \tag{4.1}$$

$$u\left(a_t, b_t; \alpha_t, \beta_t\right) = \frac{1}{1-\sigma}\left\{\left(\left[\left(\alpha_t a_t\right)^\theta + \left(\beta_t b_t\right)^\theta\right]^{1/\theta}\right)^{1-\sigma} - 1\right\}, \tag{4.2}$$

where $\sigma > 0$ and $\theta \leq 1$ are correspondingly for intertemporal and contemporary substitution. With θ as the *substitution parameter*, the elasticity of substitution is $-1/(1-\theta)$. As θ ranges from $-\infty$ to 1, absolute value of the elasticity of substitution ranges from 0 to ∞. The degree of substitution is greater than unit if $\theta \in (0, 1]$ and smaller than unit if $\theta < 0$.

The representative agent faces budget, labor, capital, and technology constraints, $\forall t \geq 0$, as follows

$$p_{at}\left(a_t + x_{at}\right) + p_{bt}\left(b_t + x_{bt}\right) = w_t\left(l_{at} + l_{bt}\right) + r_t\left(k_{at} + k_{bt}\right) \tag{4.3}$$

$$l_{at} + l_{bt} + n_{at} + n_{bt} = 1 \tag{4.4}$$

$$k_{at} + k_{bt} = k_t \tag{4.5}$$

$$k_{t+1} = (1-\delta)\, k_t + \phi x_{at}^{\nu} x_{bt}^{1-\nu} \tag{4.6}$$

$$\alpha_t = \left(1 - \eta_\alpha\right) \alpha_{t-1} + \left(n_{at}\right)^\xi, \ 0 < \eta_\alpha < 1 \tag{4.7}$$

$$\beta_t = \left(1 - \eta_\beta\right) \beta_{t-1} + \left(n_{bt}\right)^\xi, \ 0 < \eta_\beta < 1, \tag{4.8}$$

where positive (a_t, b_t) and positive (α_t, β_t) denote quantity and quality, respectively; (x_{at}, x_{bt}) are nonnegative investments towards the aggregate capital stock k_{t+1}; (k_{at}, k_{bt}) are capital services from the current stock k_t; (l_{at}, l_{bt}) are labor used for quantity production, while (n_{at}, n_{bt}) are for quality innovation; w_t is the wage rate for (l_{at}, l_{bt}); r_t is the capital rental rate; $\nu \in (0, 1)$ is the *contribution factor*; $\phi > 0$ is the *scale factor*; δ is the capital depreciation rate; and $\xi \in (0, 1)$ is the *innovation parameter*. Note that, under perfect competition, profits will be zero and play no part in the budget constraint.

Thus, besides two consumption products, there is one investment good used only for production. The law of motion for aggregate capital stock in (4.6) has the mixing feature used by, for example, Kehoe & Ruhl (2005). In addition, the agent uses "home production" to acquire know-how for improving product quality. Specifically, quality indices of a and b respectively have the laws of motion in (4.7) and (4.8). "Home production" simply means that it is the agent rather than firms who accumulates know-how which will then be transferred into product quality. For example, a teacher can revise some syllabus at his or her own will, either at home or at school.

On the quantity production side, each sector is represented by a competitive firm. The firms have Cobb-Douglas quantity production functions

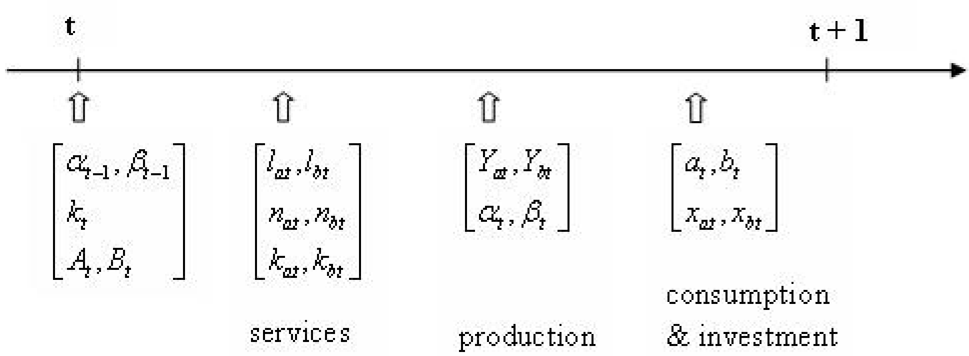
$$Y_{at} = A_t K_{at}^{\mu} L_{at}^{1-\mu},\ 0 < \mu < 1 \tag{4.9}$$

$$Y_{bt} = B_t K_{bt}^{\gamma} L_{bt}^{1-\gamma},\ 0 < \gamma < 1, \tag{4.10}$$

where (K_{at}, K_{bt}) and (L_{at}, L_{bt}) are demanded capital and labor, respectively; and strictly positive (A_t, B_t) are sectoral total factor productivity (TFP) which may follow some deterministic or stochastic processes, and have a common aggregate variation component. In our model, productivity (A_t, B_t) is the sole and ultimate exogenous source of variations.

Before explicitly laying out the timing of the economy, we have several notes. First, the effective consumption quantity is a product of physical quantity and quality index, neither of which can be zero. Second, the representative agent will not use all of labor endowment for quantity production, because he or she also wants to spend some time on quality innovation. If the agent does not spend any time on innovation, product quality depreciates as the result of forgetfulness. Third, for simplicity, we rule out the possibility of reversing investment good into consumption products. Fourth, capital services are purely quantitative, e.g. number of machines. Fifth, sectoral productivity only has direct effects on quantity production.

Figure 4.1: Timing of the economy



Let $\omega_t = \{\alpha_{t-1}, \beta_{t-1}, k_t, A_t, B_t\}$ be the information set at the beginning of period t, where $(\alpha_{t-1}, \beta_{t-1})$ denote quality of the products produced at time $t-1$; k_t is the capital stock; and (A_t, B_t) are sectoral productivity shocks. Timing of the economy in period $t \geq 0$ is as follows (Figure 4.1). First, information set ω_t is observed. Second, the representative agent decides on factor uses $\{(l_{at}, l_{bt}); (n_{at}, n_{bt}); (k_{at}, k_{bt})\}$. Third, the representative firms produce their output quantities (Y_{at}, Y_{bt}) with new quality levels (α_t, β_t) supplied by the agent. Fourth, given the factor incomes and quality levels, the representative agent decides on how much to consume and invest $\{(a_t, b_t); (x_{at}, x_{bt})\}$ to maximize the continuing expected life-time utility. This sequence makes it clear that the agent may want to spend some efforts on quality innovation. In essence, all the actions can happen simultaneously at point t.

4.2.2 A Perfectly Competitive Equilibrium

Besides primitives in preference and technology, we need rules for the interactions on markets: a perfectly competitive equilibrium. Before defining the equilibrium, we formalize the utility and profit maximization problems.

Let the extended information set in period t be $\widehat{\omega}_t = \omega_t \cup \{(\alpha_t, \beta_t)\}$; and $C_{1t} = \{(l_{at}, l_{bt}); (n_{at}, n_{bt}); (k_{at}, k_{bt})\}$, $C_{2t} = \{(a_t, b_t); (x_{at}, x_{bt})\}$ be the decisions. The representative agent maximizes the life-time expected utility following the dynamic programming problem

$$V(\omega_t) = \max_{C_{1t}, C_{2t}} \{u(a_t, b_t; \alpha_t, \beta_t) + \lambda E_{\widehat{\omega}_t} V(\omega_{t+1})\} \tag{4.11}$$

subject to the constraints in (4.3)-(4.5), the law of motion for the capital stock in (4.6), and the evolution of quality indices in (4.7)-(4.8).

On the production side, the representative firms solve the static profit maximization problems for every period t

$$\max_{\{L_{at}, K_{at}\}} \left\{p_{at} A_t K_{at}^{\mu} L_{at}^{1-\mu} - w_t L_{at} - r_t K_{at}\right\} \tag{4.12}$$

$$\max_{\{L_{bt}, K_{bt}\}} \left\{p_{bt} B_t K_{bt}^{\gamma} L_{bt}^{1-\gamma} - w_t L_{bt} - r_t K_{bt}\right\}, \tag{4.13}$$

with the necessary and sufficient conditions

$$w_t = (1-\mu)\frac{p_{at} Y_{at}}{L_{at}} = (1-\gamma)\frac{p_{bt} Y_{bt}}{L_{bt}} \tag{4.14}$$

$$r_t = \mu \frac{p_{at} Y_{at}}{K_{at}} = \gamma \frac{p_{bt} Y_{bt}}{K_{bt}}. \tag{4.15}$$

Definition 4.2.1. *A perfectly competitive equilibrium of the economy is the set of policy functions* $\{(l_{at}, l_{bt}); (n_{at}, n_{bt}); (k_{at}, k_{bt}); (a_t, b_t); (x_{at}, x_{bt})\}_{t=0}^{\infty}$ *for the representative agent,* $\{(L_{at}, L_{bt}); (K_{at}, K_{bt})\}_{t=0}^{\infty}$ *for the representative firms, prices* $\{(p_{at}, p_{bt}); (w_t, r_t)\}_{t=0}^{\infty}$*, and the associated value function* $V(\omega_t)$ *such that*

(i) given prices, policy functions are solutions to the dynamic programming problem (4.11) and the profit maximization problems (4.12)-(4.13);

(ii) given policy functions, prices clear all markets

$$a_t + x_{at} = Y_{at} \tag{4.16}$$

$$b_t + x_{bt} = Y_{bt} \tag{4.17}$$

$$L_{at} = l_{at} \tag{4.18}$$

$$L_{bt} = l_{bt} \tag{4.19}$$

$$K_{at} = k_{at} \tag{4.20}$$

$$K_{bt} = k_{bt}. \tag{4.21}$$

In equilibrium, the representative firms make zero profits, leaving the utility maximization problem (4.11) valid. Thus, the definition of a perfectly competitive equilibrium completes our baseline economy.

4.2.3 Hypothesis of Augmented Capital

In the baseline environment, quality does not have any effect on quantity production. We relax this assumption in a simple extension. Let $\overline{q}_t$ be the average quality of the investment good, and $\overline{q}_t$ follows the evolution

$$\overline{q}_{t+1} = (1 - S_q)\, \overline{q}_t + S_q \alpha_t^{\nu} \beta_t^{1-\nu},\ \ S_q \in (0, 1)\,, \tag{4.22}$$

where S_q is the weight for the addition of new quality mix. The hypothesis of augmented capital says that effective capital services embed capital quality. Specifically, the modified sectoral production functions have the forms

$$Y_{at} = A_t \left(\overline{q}_t K_{at}\right)^{\mu} L_{at}^{1-\mu} \tag{4.23}$$

$$Y_{bt} = B_t \left(\overline{q}_t K_{bt}\right)^{\gamma} L_{bt}^{1-\gamma}. \tag{4.24}$$

In reality, we do not often directly observe capital quality. However, we can observe output quantity, capital stock, and labor. Thus, a direct application of the standard growth accounting method on (4.23), for example, will generate a productivity measure of $A_t\left(\overline{q}_t\right)^{\mu}$ rather than of A_t. The question is: how does our introduction of capital quality alter the TFP estimation in the baseline model? The answer to this question may guide us in choosing the right model in empirical studies, i.e. with or without capital quality.

4.3 Productivity-Quality Causation

This section is devoted to theoretical examinations, especially the causation from productivity to quality. We first characterize the baseline economy and then deal with the augmented capital hypothesis. In the baseline economy, the equilibrium allocation is Pareto optimal. This Pareto optimal allocation can be implemented through a competitive equilibrium with some price system. Thus, to solve for the perfectly competitive equilibrium, we follow a two-step algorithm: (i) find the equilibrium allocation with a social planner problem; and (ii) given the equilibrium allocation, derive the equilibrium prices.

Definition 4.3.1. *The social planner finds the functions* $\{C_{1t}, C_{2t}\}_{t=0}^{\infty}$, *with* $C_{1t} = \{(l_{at}, l_{bt}); (n_{at}, n_{bt}); (k_{at}, k_{bt})\}$ *and* $C_{2t} = \{(a_t, b_t); (x_{at}, x_{bt})\}$, *as solutions to the dynamic programming problem*

$$V\left(\omega_t\right) = \max_{C_{1t}, C_{2t}} \left\{u\left(a_t, b_t; \alpha_t, \beta_t\right) + \lambda E_{\widehat{\omega}_t} V\left(\omega_{t+1}\right)\right\} \tag{4.25}$$

subject to the constraints

$$l_{at} + l_{bt} + n_{at} + n_{bt} = 1 \tag{4.26}$$

$$k_{at} + k_{bt} = k_t \tag{4.27}$$

$$a_t + x_{at} = A_t k_{at}^{\mu} l_{at}^{1-\mu} \tag{4.28}$$

$$b_t + x_{bt} = B_t k_{bt}^{\gamma} l_{bt}^{1-\gamma}, \tag{4.29}$$

the law of motion for aggregate capital stock in (4.6), and evolution of product quality in (4.7)-(4.8).

In Definition 4.3.1, the maximization problem has a strictly concave objective function with a convex compact constraint set. That means the social planner solution uniquely exists. Given the optimal allocation, the equilibrium wage and interest rates are derived from (4.14)-(4.15). In addition, we can pin down the product price ratio for every period t based on the condition that marginal utility-price ratios should be equated between the two sectors

$$p_{bat} = \left(\frac{\beta_t}{\alpha_t}\right)^{\theta} \left(\frac{b_t}{a_t}\right)^{\theta-1}, \tag{4.30}$$

and the budget share for b follows the identity

$$S_{bt} = \frac{p_{bt} b_t}{p_{at} a_t + p_{bt} b_t}. \tag{4.31}$$

Necessary and sufficient conditions for the optimal decisions in Definition 4.3.1 constitute a complicated system, i.e. (C.5)-(C.14) in Appendix C.

So far, we have had the social planner solution and corresponding decentralized equilibrium in a representative agent model. As $\xi \in (0, 1)$, the representative agent will spend positive amounts of time on quality innovation even though he or she does not receive compensation for accumulating know-how from firms. In our model, the innovation incentive comes from the desire for more utility in consumption. Starting from zero innovation efforts, the representative agent can always do better by spending infinitesimal amount of time because marginal gains are very large, i.e. slopes of the innovation functions in (4.7) and (4.8) are infinite at zero.

One question arises. What happens to innovation efforts if there are a continuum of agents, which is a reality, rather than a representative agent? Our concern is that the agents may free ride on the efforts of one another, and hence spend no time on quality innovation. To stop this behavior and come back the representative agent equilibrium, we need to restrict our considerations to symmetric equilibria. The argument runs as follows. First, there is no symmetric equilibrium in which all agents make zero innovation efforts. The reason is, as mentioned before, the stakes at zero efforts are so high that someone will deviate and accumulate some know-how. Second, for well-behaved preference and technology primitives, there exists a set of general equilibria including a symmetric one. All we need to do next is to select this symmetric equilibrium and come back to the representative agent case. Finally, the symmetry restriction is valid because we are currently interested in the average behavior of the economy.

4.3.1 Comparative Statics

There are several motivations behind our study of comparative statics in some steady state indexed by sectoral productivity (A, B). First, given (A, B), we can analytically solve for the steady-state equilibrium including allocations and a supporting price system which is unique up to some scale (Appendix C.2). Second, in the steady state, we can perturb productivity to learn about the productivity-quality causation and other equilibrium behaviors of interest. Third, we can identify the parametric subspace which is relevant for the steady states and later refine that for the equilibrium dynamics. In overall, this is a good starting point for exploring our economy.

Lemma 4.3.2. *Given $\Lambda = \nu(\mu - 1) + (1 - \nu)(\gamma - 1) \neq 0$, $\theta \neq \frac{1}{1+\xi}$, and some A, changes in relative productivity B/A have qualitative effects on relative consumption, quality, price, and budget share, respectively, as follows*

$$\begin{aligned} sign\left[\partial\left(b/a\right)/\partial\left(B/A\right)\right] &= sign\left[\left(1-\theta\xi\right)\left(\theta\xi+\theta-1\right)\Lambda\right] && (4.32)\\ sign\left[\partial\left(\beta/\alpha\right)/\partial\left(B/A\right)\right] &= sign\left[\theta\left(\theta\xi+\theta-1\right)\Lambda\right] && (4.33)\\ sign\left[\partial p_{ba}/\partial\left(B/A\right)\right] &= sign\left[\Lambda\right] && (4.34)\\ sign\left[\partial S_b/\partial\left(B/A\right)\right] &= sign\left[\theta\left(\theta\xi+\theta-1\right)\Lambda\right]. && (4.35) \end{aligned}$$

Proof. Appendix C.2. ■

The restrictions in Lemma 4.3.2 deserve some explanations. First, $\Lambda = (\mu-1)\nu + (\gamma-1)(1-\nu) \neq 0$. Recall that $(1-\mu)$ and $(1-\gamma)$, under perfect competition, are the labor shares respectively in sector 1 and sector 2, and ν is the contribution share in accumulating the aggregate capital stock. Thus $|\Lambda|$ is an average labor share. The restriction means that labor should play some part in the economy. Generically, $\mu \in (0,1)$, $\gamma \in (0,1)$, $\nu \in (0,1)$, and hence $\Lambda \in (-1,0)$. Second, $\theta \neq \frac{1}{1+\xi}$, or the elasticity of substitution $-1/(1-\theta)$ cannot be $-(1+\frac{1}{\xi})$. This condition guarantees that the quality ratio β/α is well defined. Recall that $\xi \in (0,1)$ is the curvature in quality evolution functions (4.7)-(4.8). In words, ξ positively governs the degree of easiness in quality innovation, and hence plays a key role in the dynamics of quality indices. When $\theta = \frac{1}{1+\xi}$, quality indices are canceled out in the system of FOCs, specifically (C.51), and become indeterminate. Besides this restriction, generically, $\theta < 1$, i.e. products are not linearly substitutable, and hence $\theta\xi < 1$. In addition, from (4.33) and (4.35), it is more interesting to have $\theta \neq 0$. Third, sectoral productivity A is fixed in Lemma 4.3.2. The reason is that, except for $\nu = 0.5$, the left-hand-side objects in (4.32)-(4.35) cannot be expressed as functions of only B/A without extra A or B terms. In the symmetry case, i.e. $\nu = 0.5$, all those objects can be defined as closed-form functions of relative productivity B/A.

In combination, $\Lambda < 0$, $\theta < 1$, $\theta \neq 0$, and $\theta \neq \frac{1}{1+\xi}$. Conditional on ξ, let $\theta < 0$ be *little substitutability*; $\theta \in (0, \frac{1}{1+\xi})$ be *medium substitutability*; and $\theta \in (\frac{1}{1+\xi}, 1)$ be *large substitutability*. There are some observations based on Lemma 4.3.2 as follows (note that every object is in relative terms).

Proposition 4.3.3. *Given some A, $\Lambda < 0$, $\theta < 1$, $\theta \neq 0$, $\theta \neq \frac{1}{1+\xi}$, and conditional on $\xi \in (0,1)$, in the steady states: (i) productivity affects consumption positively for little-medium substitutability, and negatively for large substitutability; (ii) productivity affects quality positively for medium substi-*

tutability, and negatively for little or large substitutability; (iii) productivity always has negative effects on relative price; and (iv) productivity affects budget share in the same way as quality. These results hold for all A if $\nu = 0.5$.

Thus, key links in the economy qualitatively depend on the substitution parameter θ, conditional on ξ. We have some insights for the results in Proposition 4.3.3. The initial effect of a positive productivity shock in sector 2, i.e. a surge in B, is a potential increase in supply of b. As the supply of b increases, marginal utility of this product decreases, pressing the relative price p_{ba} down. Part (iii) says this negative relative price effect prevails no matter what happens next, independent of the substitution pattern. That is a surge in B would finally make b's marginal value decrease relatively to that of a. To understand parts (i), (ii), and (iv) we consider three cases. For little substitutability, the utility function with respect to effective consumption of either a or b is quite concave. That means the agent wants to keep a close balance between $\alpha \times a$ and $\beta \times b$. As quantity b increases, the agent transfer some capital from b to produce more a and uses the freed-up labor to largely improve quality α. In this case, negative price effect dominates positive quantity effect, making the budget share for b decrease. For medium substitutability, the agent does not need to keep a close balance between effective consumptions of a and b. In addition, as effective consumption is a product between quantity and quality, the agent benefits the most by improving both quantity and quality of largely one sector. As quantity b increases, it is optimal for the agent to improve quality β. In this case, positive response in demand for b overwhelms reduction in p_{ba}, leading to a larger S_b. For large substitutability, again, the agent only needs to work mostly on one sector. As products are now very easy to be substituted, the freed-up resources after a surge in B will be spent mostly on quantity and quality of product a. As the agent demands more a relative to b, and with a decrease in p_{ba}, the budget share for b decreases.

Clearly, ξ and θ play important roles in what to produce, what to consume, and what to improve upon. In empirical studies, these parameters should be fitted to data. Within some ball around the steady state, we expect that those results in (4.32)-(4.35) would hold for equilibrium dynamics.

4.3.2 Equilibrium Dynamics

With a numerical exercise, we look at equilibrium dynamics of the economy in the neighborhood of some steady state, i.e. an RBC model. For simplicity, the equilibrium is approximated by linear laws of motion of endogenous variables. The exercise will generate correlation coefficients shedding light on the economic relations of interest. We use the parameter values in Table 4.1 for simulations. The processes governing (A_t, B_t) will be specified shortly.

Table 4.1: Baseline parameter values in simulations

description	symbol	range	value
curvature of CES	θ	$(-\infty, 1]$	$-3; 0.3; 0.7$
time discount factor	λ	$(0, 1)$	0.95
curvature of CRRA	σ	$(0, \infty)$	2.0
capital depreciation rate	δ	$(0, 1)$	0.05
investment contribution factor	ν	$(0, 1)$	0.45
investment scale factor	ϕ	$(0, \infty)$	1.0
capital share in a production	μ	$(0, 1)$	0.4
capital share in b production	γ	$(0, 1)$	0.35
depreciation of α	η_α	$(0, 1)$	0.1
depreciation of β	η_β	$(0, 1)$	0.12
quality innovation parameter	ξ	$(0, 1)$	2/3

There are reasons behind the parameter choice. First, as noted earlier, conditional on ξ, the substitution parameter plays an important role in the relations between variables. We define $\xi = 2/3$, which means the threshold $\frac{1}{1+\xi} = 0.6$, and examine three specific values of θ corresponding to little, medium, and large substitutability. Specifically, the corresponding θ values are -3, 0.3, and 0.7. Second, the subjective discount rate is assumed to be 5.2 percent, and the discount factor is $\lambda = 0.95$. Third, the constant rate of risk aversion is $\sigma = 2$. In fact, σ does not play any role in comparative statics. However, it critically governs intertemporal decisions in equilibrium dynamics. Fourth, the capital depreciation rate is assumed to be 5 percent per year. Fifth, it is more interesting to look at sectoral asymmetry, i.e. we have different capital shares and quality depreciation rates for the two sectors. Finally, for simplicity, $\phi = 1$.

The time path $\{A_t, B_t\}_{t=1}^{T}$ is generated according to a VAR model

$$\begin{bmatrix} A_t \\ B_t \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.5 \end{bmatrix} + \begin{bmatrix} 0.7 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} A_{t-1} \\ B_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{at} \\ \varepsilon_{bt} \end{bmatrix}, \tag{4.36}$$

where $A_1 = 1$; $B_1 = 1$; and $\varepsilon_t = [\varepsilon_{at}\ \varepsilon_{bt}]' \overset{i.i.d.}{\sim} N(\mathbf{0}, \Sigma)$ with

$$\mathbf{\Sigma} = \begin{bmatrix} 2.04 \times 10^{-4} & 0 \\ 0 & 4.6875 \times 10^{-4} \end{bmatrix}. \tag{4.37}$$

Effectively, $A_t \sim N(1, 0.02^2)$, $B_t \sim N(1, 0.025^2)$, and they are not correlated. For simplicity, we do not allow A_t and B_t to be correlated.

As some key variables can be observed in reality, model correlation patterns can be matched with data counterparts for clues on the range of the parameters ξ and θ. Recall that decisions are functions of the information set ω. We first log-linearize the system of FOCs, and then impose the laws of motion $z_t = \Gamma z_{t-1} + \Psi s_t$, where z_t includes key endogenous variables and s_t contains the exogenous productivity shocks (Appendix C.3). In the log-linearization transformation, the variables $\{z_t, s_t\}$ are interpreted as percentage deviations from the nonstochastic steady state counterparts. As soon as transformed data are simulated, we can back out the original variables and generate correlation coefficients. The process that generates s_t is derived from (4.36) and (4.37). The feedback and feedforward matrices Γ and Ψ are solved with the method of undetermined coefficients by Christiano (2002).

In this exercise, eigenvalues of Γ for $\theta = \{-3, 0.3\}$ are strictly less than unit, keeping the dynamic systems stable. However, the dynamic system becomes unstable for $\theta = 0.7$. In fact, for $\theta \in (0.6, 1)$, the system is not stable most of the time; and if it is stable, quality indices may have negative values. For $\theta < 0.6$, e.g. $\theta = 0.59$, the system is stable (Appendix C.3). In general, by varying ξ and θ, we see that the system is always stable for $\theta < \frac{1}{1+\xi}$, and it becomes chaotic most of the time for $\theta > \frac{1}{1+\xi}$. In words, too much substitution between products a and b does not produce stable and sensible dynamics. With no friction in factor mobility, productivity variations may cause too much turnover of labor and capital between product sectors.

Table 4.2: Dynamics: correlation patterns

line	description	$\theta < 0$	$\theta \in (0, \frac{1}{1+\xi})$
1.	B/A & Y_b/Y_a	+	+
2.	B/A & b/a	+	+
3.	B/A & β/α	−	+
4.	B/A & p_{ba}	−	−
5.	B/A & S_b	−	+
6.	B/A & $(l_b + n_b)/(l_a + n_a)$	−	+
7.	B/A & l_b/l_a	−	+
8.	B/A & n_b/n_a	−	+
9.	B/A & k_b/k	−	+
10.	Y_{bt}/Y_{at} & b_t/a_t	+	+
11.	Y_{bt}/Y_{at} & p_{bat}	−	−
12.	b_t/a_t & p_{bat}	−	−
13.	β_t/α_t & p_{bat}	+	−

Note: θ for substitution; ξ for innovation.

Thus, we further impose that $\theta < \frac{1}{1+\xi}$ for $\xi \in (0, 1)$. What do we have with this restriction? The correlation patterns turn out to be consistent with the results in Proposition 4.3.3 (Table 4.2). First, as expected, the correlation between B/A and p_{ba} does not depend on θ. Second, the correlation between Y_b/Y_a and b/a is always positive, e.g. if relative output is higher, the corresponding relative consumption also increases. In addition, Y_b/Y_a and b/a have the same correlation patterns with B/A, or p_{ba}, conditional on θ. This implies that, in empirical studies, we only need data on either Y_b/Y_a or b/a. Third, the correlation between relative productivity and quality lines up with Proposition 4.3.3. Specifically, a positive relative productivity shock decreases relative quality for little substitutability, and the reverse holds for medium substitutability. This is also the case in the relationship between B/A and S_b. Fourth, for little substitutability or $\theta < 0$, relative quality and relative price have positive correlation. This result differs from that in Chapter 3 even though the two models arrive at quite similar relative price functions as in (4.30). The reason for the difference lies in the nature of the relationship between relative quality and relative price. Chapter 3 looks at a direct relationship between relative quality and relative price, in which the former is exogenous and causes the latter. Thus, for $\theta < 0$, relative quality and relative price have a negative correlation. In the current study, both relative quality and relative price are endogenously driven by the same exogenous productivity shocks, and have positive correlation when θ is negative. In addition, the difference also happens for $\theta \in (0, 0.5)$. In reality, if quality has some exogenous components which dominate the effects driven by productivity shocks, we may again see the results in Chapter 3.

Lines 6-9 of Table 4.2 also support our predictions about how the agent reallocate labor and capital when facing a positive shock in relative productivity B/A. For little substitutability, to keep a close balance between effective consumptions, the agent transfers some labor and capital from sector 2 to sector 1, reducing relative quality β/α. Note that relative output Y_b/Y_a still rises for a dominant increase in B/A. For medium substitutability, there is not much difference between the two products. Thus, an increase in B/A stimulates a diversion of resources from sector 1 to sector 2, raising β/α.

4.3.3 Does Capital Quality Matter?

The extended model, in which capital quality plays some role in production functions, is presented in Appendix C.4. We examine the equilibrium dynamics of this model. In fact, the introduction of capital quality does not really affect growth accounting because in all cases $\overline{q}_t$ virtually does not change over time. In other words, for example, the time series A_t and $A_t(\overline{q}_t)^{\mu}$ are nearly the same after rescaling. The direct reason is that no matter how the ratio β_t/α_t evolves, individual quality indices tend to move in opposite directions. e.g. when β_t increases, α_t decreases. The intuition is that if changing the ratio β_t/α_t is one instrument for utility maximization, then moving innovation labor between α_t and β_t is a direct way. As individual quality indices move in opposite directions, aggregate quality, which is a quality mixture (4.22), should not change much over time.

We have characterized the equilibrium dynamics in an RBC model. How about a growth model? Our established results remain valid. We embed considerations of a growth model in the upcoming US application.

4.4 Application to US Services-Goods Economy

Based on the previous discussion, we choose the baseline model to be fitted to the services and goods sectors of the US economy. Sector 1 produces goods (a), and sector 2 offers services (b). The ultimate exogenous driving force in the economy is productivity evolution in the two sectors. The object of our main interest is services-goods relative quality driven by productivity changes. There are three specific tasks. First is finding the parameter values so that the numerical model can generate certain moments in the data trends. Second is perturbing the productivity shocks around the trends to see if the model can mimic the patterns in data time paths and infer endogenous quality innovation. Third is comparing productivity-driven quality innovation with total quality innovation, which is based on Chapter 3. Before carrying out the tasks, we examine the data.

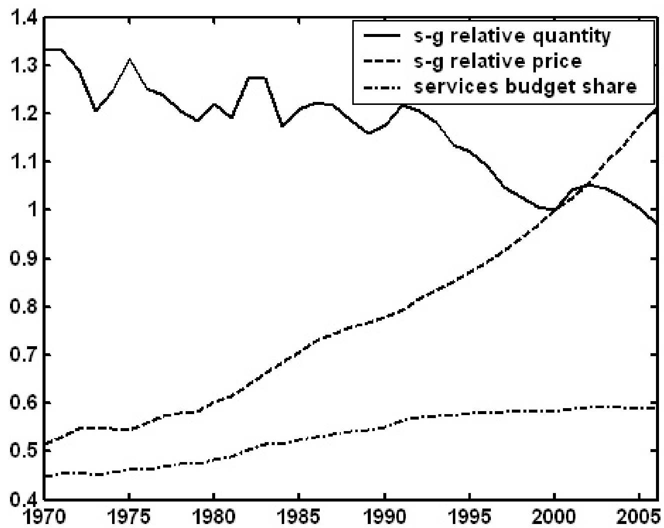
4.4.1 Data Description

Our US data come from the national income and product accounts (NIPA) by the Bureau of Economic Analysis (BEA). All the data are publicly available and can be readily downloaded from the BEA's Web site. Postwar data are available from 1946 to 2006. However, only the 1970-2006 data are used. The horizon restriction is based on the fact that time series in 1970-2006 have clear trends which are useful to the upcoming moment matching exercise.

Classifications of goods and services follow the definitions of NIPA tables. The broad components of goods industries are agriculture, forestry, and fisheries; mining; and manufacturing. The services industries are transportation and public utilities; wholesale trade; retail trade and automobile services; finance, insurance, and real estate; different services; and government services. We omit residential and non-residential structures because they are composed of mixed quality levels and can render services for a very long period of time.

The original time series and their corresponding source NIPA tables are: (i) goods quantity index Y_{at} (1.2.3); (ii) goods price index P_{at} (1.2.4); (iii) services quantity index Y_{bt} (1.2.3); (iv) services price index P_{bt} (1.2.4); and (v) budget share for services S_{bt} (1.5.5). Besides these key time series, there are other data pieces which will be specified later. All the quantity and price time series are first normalized so that their indices equal unit in the year 2000. The services-goods relative price is constructed as P_{bt}/P_{at}. Individual prices may have a lot of noises like inflation and taxes. However, the price ratio is supposed to largely bear relative quantity and quality information (Chapter 3). By the same token, the budget share for services should evolve mostly under relative quantity and quality effects. It is noted that, reliable measures of labor allocation in quantity production and quality innovation are not available. This means we cannot use the standard growth accounting exercise to estimate sectoral productivity evolutions.

Figure 4.2: US services-goods economy, 1970-2006



Source: NIPA tables (BEA).

There are some key data features. First, goods and services quantities have upward trends with relatively stable growth rates (Figures 4.3 and 4.4). Second, quantity of goods grows more quickly than that of services, leading to a downward trend in the services-goods relative quantity time path (Figure 4.2). Third, the services-goods relative price and the budget share for services are increasing over time. As predicted by the model, relative quantity and price are always negatively correlated. The opposite trends in the services relative quantity and budget share suggest that goods and services have little substitutability, i.e. $\theta < 0$, which seems intuitive for this product dichotomy.

In combination, the tasks have to rely on a limited data set. In addition, the economy we are trying to match is not the original one, but a normalized version of that. This normalization may cause inconsistencies between different objects. Next, we will discuss how to deal with these problems.

4.4.2 Parameter Values

The task is to numerically specify the baseline model. Among the parameters, $\{\theta, \eta_\alpha, \eta_\beta, \xi\}$ are of our main interest because they critically govern the (relative) quality innovation process. In the literature, many studies rely on detrended data to estimate parameters. In this application, we do not rely on detrended data because the relative price and budget share concepts prevent us from doing so. In addition, we cannot apply GMM estimation methods based on the FOC system because productivity, detailed labor and capital uses, and quality indices are not directly observed. Our strategy is to first specify uncritical parameters and then use moment matching to pinpoint the four mentioned above. Parameter values are summarized in Table 4.3.

Table 4.3: Parameter values in US application

description	symbol	value	source
curvature of CES	θ	-7.0	matching
time discount factor	λ	0.97	assigning
curvature of CRRA	σ	2.0	assigning
capital depreciation rate	δ	0.06	assigning
investment contribution factor	ν	0.42	assigning
investment scale factor	ϕ	1.0	assigning
capital share in a production	μ	0.32	assigning
capital share in b production	γ	0.34	assigning
depreciation of α	η_α	0.0045	matching
depreciation of β	η_β	0.0055	matching
quality innovation parameter	ξ	0.69	matching

Using the original (normalized) data means that we need to solve the nonstationary equilibrium of a growing economy. For this purpose, additional assumptions are imposed on the model and data. First, for simplicity, the agent is assumed to have correct expectations about future productivity evolution in both sectors. In the model, output growth comes exclusively from productivity changes. Thus, for the sample horizon 1970-2006, projections of $\{A_t, B_t\}$ are based on the time series $\{Y_{at}, Y_{bt}\}$. The model is effectively deterministic because the distribution of future states is degenerate. Second, we need to specify the initial and terminal conditions. The initial state composed of total capital stock and quality indices is assumed to take on steady state values if the 1970's sectoral productivity levels stay forever. For the terminal conditions, we assume that the economy goes on for 10 more years after 2006. It is ideal to have a very long model horizon to guarantee that the terminal conditions do not affect behaviors in the sample horizon. However, this truncation of future is necessary for feasible computations and reliable projections of productivity evolutions. Technically, if the out-of-sample horizon is too long, the two projected sectoral productivity time paths diverge so much that numerical computations of the equilibrium are not reliable. In fact, to compute the equilibrium, we implement the Newton's method in C++ with the initial guess at steady state values of each year (Appendix C.5).

Uncritical Parameters. From Proposition 4.3.3, we know that some parameters are not important to the relative performance between the sectors. Those uncritical in the (relative) quality innovation process are $\{\lambda,\sigma,\delta,\nu,\phi, \mu,\gamma\}$. First, the time discount factor λ and capital depreciation rate δ are rounded-off values based on the US calibration exercise by Cooley & Prescott (1995). Second, the CRRA curvature parameter σ is often chosen to be unit so that the utility function is logarithmic. We choose another value for this parameter to avoid any special effects possibly linked to the logarithmic form. Third, specification of the contribution factor is based on a derivation of the FOC that $\nu = p_{at}x_{at}/(p_{at}x_{at} + p_{bt}x_{bt})$. In words, ν is the share of sector 1 in investment market value. This parameter should not be too far from the budget share for goods in consumption, which is 0.47 on average in 1970-2006 (NIPA 1.5.5.). This numerical value is not compatible with the computation of the equilibrium. Via experiments, we fix ν at 0.42, i.e. goods contribute a smaller share than services in capital accumulation. This at first seems contradictory to the common sense that services cannot be accumulated. However, it should be borne in mind that almost all economic activities have some services contents like electricity, transportation, insurance, and finance. In general, these various services play a crucial role in production, location, purchases, and consumption. Hence services are embodied in the capital stock. Fourth, again, we assume the investment scale factor to be unit for simplicity. Fifth, in principle, the parameters μ and γ can be specified based on income shares (NIPA 6.1). In the data, capital shares in goods and services industries are respectively 0.28 and 0.35 in 1970-2000, which means sector 2 (services) employs relatively more labor than sector 1 (goods). However, the difference between those two values is large enough to spoil equilibrium computations. The economy-wide capital share is 0.33. Based on this, we choose $\mu = 0.32$ and $\gamma = 0.34$.

Critical Parameters. Seeking $\{\theta, \eta_\alpha, \eta_\beta, \xi\}$ is an interesting exercise because they are either new in the literature or mysterious in this application. As mentioned earlier, we rely on moment matching to pin them down, where the objective function is the sum of squared differences between data and model moments. Unfortunately, as the data set is limited, we do not have enough moments to identify all of these parameters at the same time. Experiments on the discretized parameter space show that η_α and η_β should be small but cannot be zero, and $\eta_\beta > \eta_\alpha$. We assume that $\eta_\beta = 0.0055$ and $\eta_\alpha = 0.0045$, i.e. quality of services is easier to depreciate than quality of goods.

Here are some details about the moments. We specify the time paths of A_t and B_t as smooth trends based on those of Y_{at} and Y_{bt}. In addition, growth in B_t is then scaled down by a factor of 0.98 to make the model budget share for services close to the data counterpart in equilibrium. Effectively, we fix the time series of B_t/A_t. The moments reflect responses of endogenous relative objects to the evolution of B_t/A_t. The first moment is the angle between linear trends of Y_{bt}/Y_{at} and S_{bt}. The second moment is the slope of the linear trend in S_{bt}. Changes in θ and ξ do vary these moments. There is a subtle and negligible difference between data and model moments: data moments are based on linear projections of the original data, whereas model moments are based on linear projections of trends in the original data.

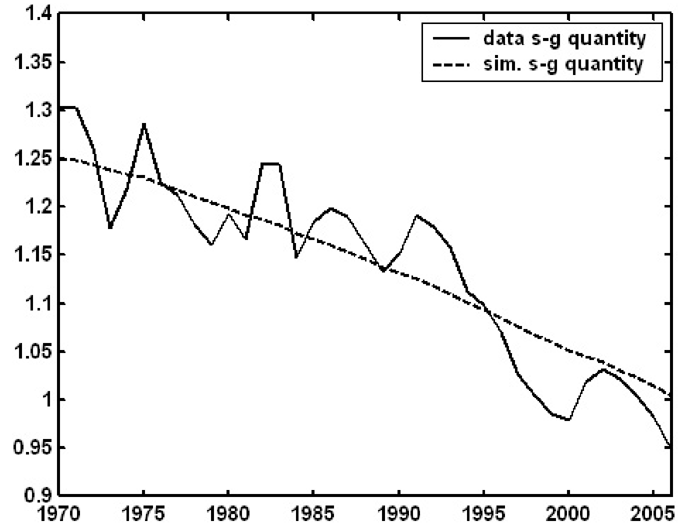
Examinations of the objective function reveal that we do not have a globally convex programming problem. This prevents us from using the simulated method of moments to estimate these two parameters. To overcome this challenge, the exercise has two steps. First is finding a good guess, and second is iteratively searching for θ and ξ. We discretize the parameter space to have guesses of θ and ξ. In fact, the final parameters are not far from the guesses. Specifically $\theta = -7.0$, $\xi = 0.69$, and note that $\theta < 1/(1+\xi)$. The corresponding elasticity of substitution between goods and services is -0.125. This means goods and services are hard to be substituted.

4.4.3 Simulated Time Paths

To pin down parameters, we rely on data trends and specify smooth time paths of A_t and B_t. Here, we perturb the time series of A_t and B_t around the smooth trends to mimic the evolution of actual Y_{at} and Y_{bt}; then generate some time paths of interest; and compare those with data. In overall, simulated objects match the data trends quite well (Figures 4.3-4.6). However, there are some features in the data that the model does not mimic satisfactorily.

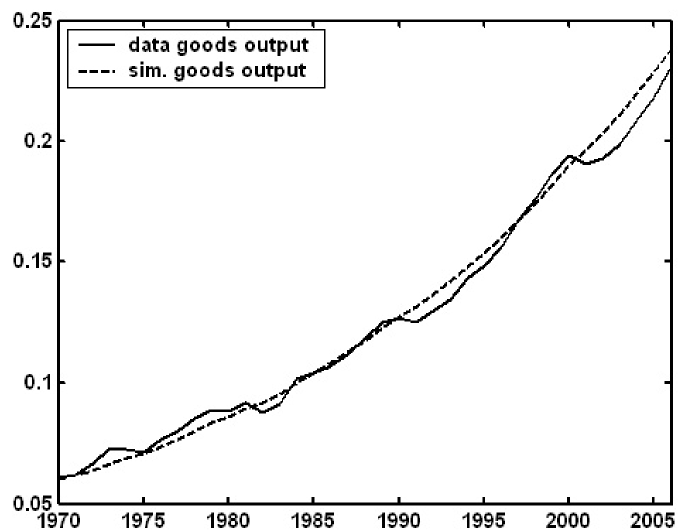
Here are some detailed observations. First, our projections of the productivity time paths can generate the extent of economic growth in 1970-2006

Figure 4.3: Services-goods relative quantity, 1970-2006



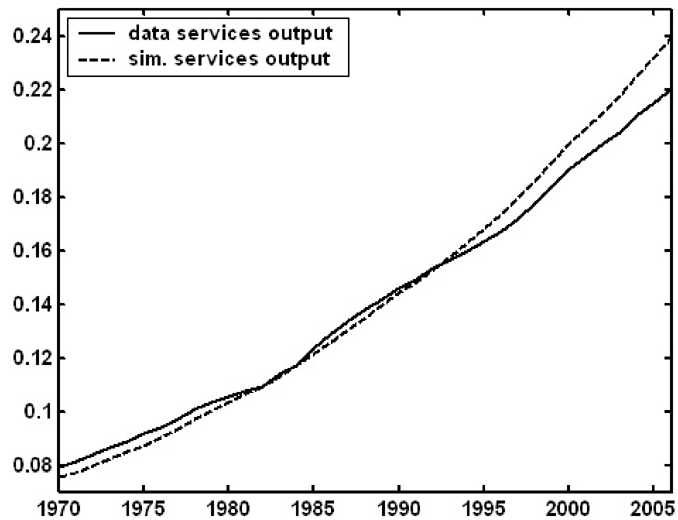
Note: simulated quantity is renormalized for comparisons.

Figure 4.4: Data and simulated goods quantity, 1970-2006



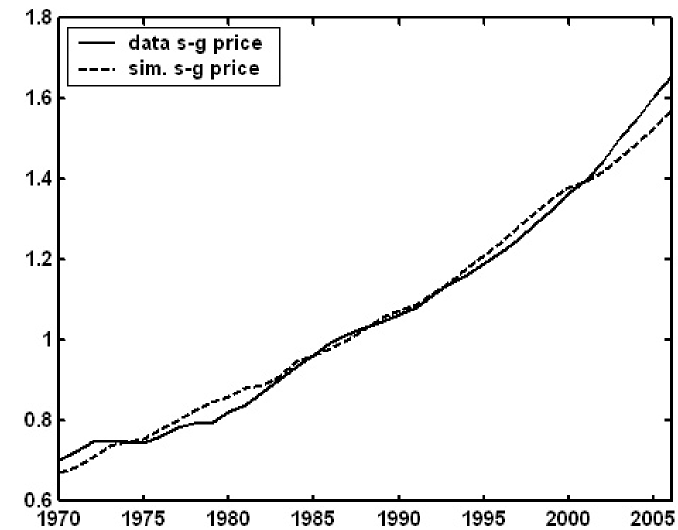
Note: data output is renormalized for comparisons.

Figure 4.5: Data and simulated services quantity, 1970-2006



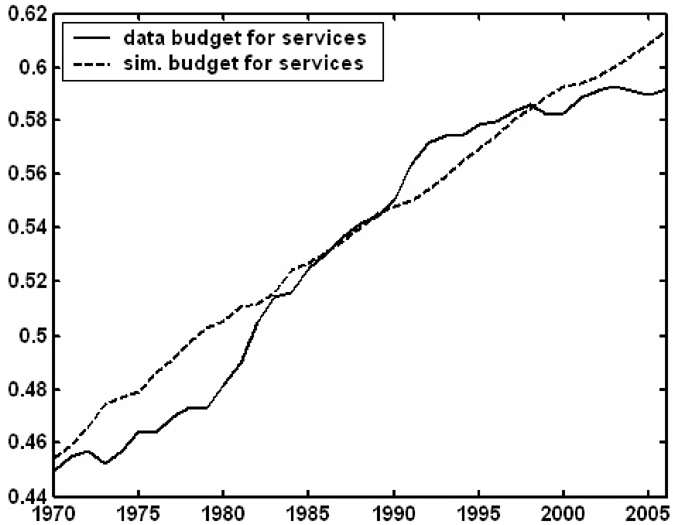
Note: data output is renormalized for comparisons.

Figure 4.6: Services-goods relative price, 1970-2006



Note: simulated price is renormalized for comparisons.

Figure 4.7: Budget share for services, 1970-2006



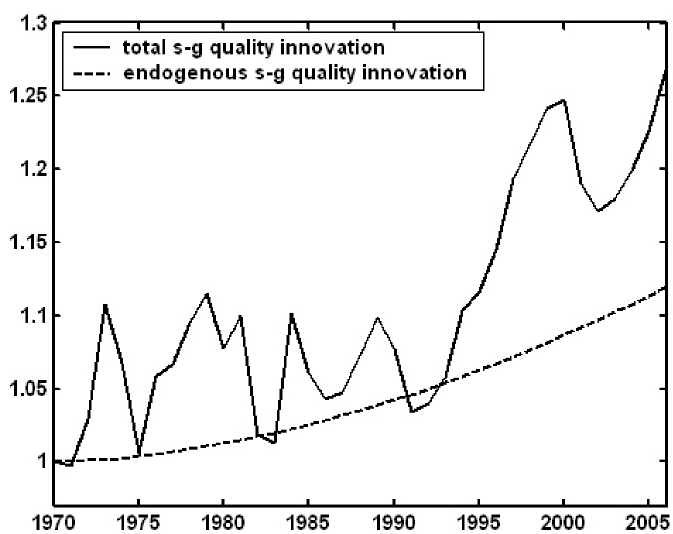
data (Figures 4.4 and 4.5). In addition, the simulated evolution patterns in output can mimic the reality. However, as the perturbations around productivity trends are small, some simulated time paths are smoother than the data evolutions like those in Figures 4.3 and 4.4. Second, the simulated services-goods relative price matches the data in both trend and pattern to a large extent (Figure 4.6). This again supports the productivity projections based on actual output time series. Third, the trend in simulated budget share for services does match the reality (Figure 4.7). However, even though the simulated and data time series have large positive correlation (0.98), their patterns are different. Specifically, while the data time series has the sigmoid shape, the simulated one shows a clear positive trend. In addition, the latter is less volatile than the former.

In overall, simulated variables have the same correlation patterns presented in Table 4.2 (for little substitutability). An increase in productivity B/A will induces a transfer of labor and capital to sector 1. Thus, relative productivity and relative quality are negatively correlated.

4.4.4 Endogenous vs. Total Quality Innovation

The previous comparisons show that the model can mimic the extent of growth in observable objects. This implies we can generate a reliable trend of services relative quality. By comparing the extent of growth in (relative) endogenous and total quality measures, we learn about the importance of the former as a macroeconomic response for welfare improvement. Recall that total quality innovation is based on the quality inference method in Chapter 3, which can be interpreted as an accounting exercise. Specifically, quantity changes alone cannot fully explain evolution of the relative price of services, and the remaining part is caused by total quality innovation. Thus, total quality innovation varies due to all possible sources like randomly developed ideas for quality improvement as well as productivity variations.

Figure 4.8: Endogenous vs. total quality innovation, 1970-2006



Note: $\theta = -7.0$; total quality innovation is based on Chapter 3; 1970's values are normalized to 1; we compare the trends of two time series.

In Figure 4.8, both relative quality concepts move upwards over time, i.e. quality of services increases in relation to that of goods. However, total quality grows faster than endogenous quality. When comparing the trends, from 1970-2006, endogenous quality accounts for about two fifths to one half of total quality growth. Thus, productivity evolution and its quality responses do play a significant role in general quality development. This also means we can see how important other factors, which are not tied to productivity as in the baseline model, collectively are. At the aggregate level, those other factors including R&D can be treated as exogenous and introduced into the laws of motion (4.7) and (4.8). We do not attempt this extension here.

4.5 Conclusion

In this study, we examine how productivity affects quality in the context of competitive growth and RBC models and how important productivity-driven quality is in total quality of the US. Differing from the variety-growth and quality-ladder literature, this study does not use market power to explain why quality varies. Specifically, labor is used for both quantity production and quality innovation. As quantity and quality can be substitutable, a change in productivity induces a reallocation of labor, leading to quality variations.

This study finds that productivity's effect on quality depends on two key parameters, which govern how substitutable the products are and how easy it is to improve quality. Specifically, productivity and quality have a negative correlation for low-range substitutability and a positive correlation for medium-range substitutability, where the upper bound of the medium range negatively depends on the innovation parameter. The model is then applied to the goods and services sectors of the US from 1970-2006 using aggregate data. The main empirical result suggests that productivity-driven quality can play a significant role in total quality. In addition, the parameter estimates imply a negative correlation between productivity and quality.

Appendix A

Derivations for Chapter 2

A.1 Expected Innovation Step Function

Given any effort $\lambda \in [0,1]$, the expected innovation step is $s_E(\lambda) = \lambda s(\lambda)$. Observe that $s_E'(\lambda) = s(\lambda) + \lambda s'(\lambda) > 0$. Consider the second-order derivative

$$s_E''(\lambda) = 2s'(\lambda) + \lambda s''(\lambda)$$

$$\iff s_E''(\lambda) = s'(\lambda)\left\{2 + \lambda s''(\lambda)/s'(\lambda)\right\}.$$

This equation suggests that we choose $s(\lambda) = \lambda^{1-\nu}$, where $\nu < 1$ is the constant rate of risk aversion. Thus, $s_E''(\lambda) = s'(\lambda)(2-\nu) > 0$, and the expected step is convex in effort.

A.2 Properties of $b(\theta_x, p_x)$

To use the implicit function theorem, based on the common FOC, we construct the following function: $F(b, \theta_x, p_x) = (\theta_x/p_x)u'(\theta_x b/p_x) - 1$. With some shorthand notations

$$F_b = \left(\frac{\theta_x}{p_x}\right)^2 u'' < 0$$

$$F_\theta = \frac{1}{p_x}u' + \frac{\theta_x b}{p_x^2}u'' = \frac{u'}{p_x}(1 - r_R) > 0$$

$$F_p = -\frac{\theta_x}{p_x^2}u' - \frac{\theta_x^2 b}{p_x^3}u'' = -\frac{\theta_x u'}{p_x^2}(1 - r_R) < 0.$$

where $r_R = -(\theta_x b/p_x)u''(\theta_x b/p_x)/u'(\theta_x b/p_x)$. Note that $\partial b/\partial\theta_x = -F_\theta/F_b$ and $\partial b/\partial p_x = -F_p/F_b$. Thus $\partial b/\partial\theta_x > 0$ and $\partial b/\partial p_x < 0$. This also implies that

$\partial b/\partial k_x = (\partial b/\partial \theta_x)(\partial \theta/\partial k_x) > 0$. We need these conditions hold unambiguously, and the assumption $r_R = \sigma \in [\frac{1}{2}, 1)$ is sufficient. In fact, based on the original FOC

$$b(\theta_x, p_x) = \left(\frac{\theta_x}{p_x}\right)^{(1-\sigma)/\sigma}. \tag{A.1}$$

A.3 Monopoly Pricing

The monopoly solves $\max_{p_x}\{b(p_x)(1 - w/p_x)\}$. Based on Appendix A.2, the FOC and its equivalent forms are

$$\begin{aligned} & b'(p_x)(1 - w/p_x) + b(p_x) w/p_x^2 = 0 \\ \Longleftrightarrow\ & -(F_p/F_b)(1 - w/p_x) + b(p_x) w/p_x^2 = 0 \\ \Longleftrightarrow\ & -\frac{(1-\sigma) b}{\sigma p_x}\left(1 - \frac{w}{p_x}\right) + \frac{bw}{p_x^2} = 0 \\ \Longleftrightarrow\ & p_x = \frac{w}{1-\sigma}. \end{aligned} \tag{A.2}$$

Plug this result into (A.1), the optimal budget share for innovation products and the monopoly profit are

$$b = \left(\frac{(1-\sigma)\theta_x}{w}\right)^{(1-\sigma)/\sigma} \tag{A.3}$$

$$\Pi^M(k_x) = \sigma\left(\frac{(1-\sigma)\theta_x}{w}\right)^{(1-\sigma)/\sigma}. \tag{A.4}$$

A.4 Decreasing Optimal Investments

Consider the equality in equation (2.10), i.e. $\lambda_x > 0$. We define

$$F(k_x, \lambda_x) = -c'(\lambda_x) + \beta\left[\left(V^M\left(k_x^+\right) - V^M(k_x)\right) + \lambda_x \Pi_1^M\left(k_x^+\right) s'(\lambda_x)\right].$$

By the implicit function theorem, $\lambda_x'(k_x) = -F_1(k_x, \lambda_x)/F_2(k_x, \lambda_x)$. Observe that at the optimal λ_x, the SOC of the Bellman equation is $F_2(k_x, \lambda_x) < 0$.

In addition, by the envelop theorem, $V_1^M(k_x) = \Pi_1^M(k_x)$. Next, we have

$$F_1(k_x, \lambda_x) = \beta\left[\left(\Pi_1^M\left(k_x^+\right) - \Pi_1^M(k_x)\right) + \lambda_x s'(\lambda_x)\Pi_{11}^M\left(k_x^+\right)\right].$$

As $\Pi^M(k_x)$ is a concave function, $\Pi_1^M(k_x^+) < \Pi_1^M(k_x)$ and $\Pi_{11}^M(k_x^+) < 0$. In combination, $F_1(k_x, \lambda_x) < 0$, and hence $\lambda_x'(k_x) < 0$.

A.5 Duopoly Investment

The Bellman equation in (2.18) can be rewritten as

$$V^D(k_x, k_y) = \max_{\lambda_x \geq 0}\left\{\Pi^D(k_x, k_y) - c(\lambda_x) + \beta\left[\begin{array}{c} \lambda_x\lambda_y V^D(k_x^+, k_y^+) + \\ \lambda_x(1-\lambda_y)V^D(k_x^+, k_y) + \\ (1-\lambda_x)\lambda_y V^D(k_x, k_y^+) + \\ (1-\lambda_x)(1-\lambda_y)V^D(k_x, k_y) \end{array}\right]\right\}.$$

The FOC, where equality holds if $\lambda_x > 0$, for this problem is

$$-c'(\lambda_x) + \beta\left[\begin{array}{c} \lambda_y V^D(k_x^+, k_y^+) + \lambda_x\lambda_y V_1^D(k_x^+, k_y^+)s'(\lambda_x) \\ +(1-\lambda_y)V^D(k_x^+, k_y) + \lambda_x(1-\lambda_y)V_1^D(k_x^+, k_y)s'(\lambda_x) \\ -\lambda_y V^D(k_x, k_y^+) \\ -(1-\lambda_y)V^D(k_x, k_y) \end{array}\right] \leq 0$$

$$\iff -c'(\lambda_x) + \beta\left[\begin{array}{c} \lambda_y\left\{\begin{array}{c} \left[V^D(k_x^+, k_y^+) - V^D(k_x, k_y^+)\right] \\ +\lambda_x V_1^D(k_x^+, k_y^+)s'(\lambda_x) \end{array}\right\} \\ +(1-\lambda_y)\left\{\begin{array}{c} \left[V^D(k_x^+, k_y) - V^D(k_x, k_y)\right] \\ +\lambda_x V_1^D(k_x^+, k_y)s'(\lambda_x) \end{array}\right\} \end{array}\right] \leq 0.$$

Finally, by the envelop theorem, the FOC becomes

$$-c'(\lambda_x) + \beta\left[\begin{array}{c} \lambda_y\left\{\begin{array}{c} \left[V^D(k_x^+, k_y^+) - V^D(k_x, k_y^+)\right] \\ +\lambda_x \Pi_1^D(k_x^+, k_y^+)s'(\lambda_x) \end{array}\right\} \\ +(1-\lambda_y)\left\{\begin{array}{c} \left[V^D(k_x^+, k_y) - V^D(k_x, k_y)\right] \\ +\lambda_x \Pi_1^D(k_x^+, k_y)s'(\lambda_x) \end{array}\right\} \end{array}\right] \leq 0, \qquad \text{(A.5)}$$

where equality holds if $\lambda_x > 0$.

A.6 Comparisons of Social Welfare

(i) WTS: $\mathbf{W}^S(k_x) > \mathbf{W}^M(k_x)$ $\forall \mathbf{k_x}$. Assume that the economy starts at k_x at time t and follows some feasible policy $\lambda^t = \{\lambda_j\}_{j=t}^{\infty}$. The existence of a threshold k^* implies that there is a time T far into the future beyond which both the monopoly and social planner do not innovate, i.e. $\lambda_j = 0$ $\forall j > T$. This means that dynamic values are finite. Formally, dynamic values are redefined as follows

$$W^M\left(k_x, \lambda^t\right) = E_{\lambda^t}\left\{\sum_{j=0}^{T-t} \Phi^{GM}(k_{x,t+j}) + \frac{\Phi^{GM}(k_{x,T+1})}{1-\beta}\right\},$$

$$W^S\left(k_x, \lambda^t\right) = E_{\lambda^t}\left\{\sum_{j=0}^{T-t} \Phi^{GS}(k_{x,t+j}) + \frac{\Phi^{GS}(k_{x,T+1})}{1-\beta}\right\},$$

where $\Phi^{GM}(\cdot)$ and $\Phi^{GS}(\cdot)$ are gross welfare flows. Given that both the monopoly and social planner follow the same λ^t, they will land on the same know-how stocks in all future paths. It is already established that $\Phi^{GS}(k_x) > \Phi^{GM}(k_x)$ $\forall k_x$. Thus $W^S(k_x, \lambda^t) > W^M(k_x, \lambda^t)$. Further assume that λ^t is optimal for the monopoly. As B is large, λ^t is also feasible for the social planner. Consequently, the social planner does strictly better by just mimicking the monopoly, and even better by carrying out the optimal policy.

(ii) WTS: $\lambda^S(\mathbf{k}_x) \geq \lambda^M(\mathbf{k}_x)$ $\forall \mathbf{k}_x$. Our claim readily holds in two non-mutually exclusive: for $t > T$, no one innovates; and $\lambda^M(k_x) = 0$ for some k_x. Consider all k_x where $\lambda^M(k_x) > 0$ and the corresponding FOC is

$$c'\left(\lambda_x\right) = \beta\left[\left(V^M\left(k_x^+\right) - V^M\left(k_x\right)\right) + \lambda_x \Pi_1^M\left(k_x^+\right) s'\left(\lambda_x\right)\right].$$

If the social planner chooses λ_x, the marginal cost is also $c'(\lambda_x)$; The marginal benefit is

$$\beta\left[\left(V^S\left(k_x^+\right) - V^S\left(k_x\right)\right) + \lambda_x U_1^S\left(k_x^+\right) s'\left(\lambda_x\right)\right].$$

Note that $\Pi^M(k_x) = [(1-\sigma)^{1/\sigma}]U^S(k_x)$, which implies $U_1^S(k_x) > \Pi_1^M(k_x)$ for $\sigma \in [\frac{1}{2}, 1)$. Further, by the envelop theorem, $V_1^S(k_x) = U_1^S(k_x)$ and $V_1^M(k_x) = \Pi_1^M(k_x)$. By integration, $V^S(k_x^+) - V^S(k_x) > V^M(k_x^+) - V^M(k_x)$. Thus at λ_x, marginal benefit is strictly larger than marginal cost for the society. As these functions are continuous in λ_x, the social planner can always find some $\varepsilon > 0$ such that marginal benefit is still larger than marginal cost at $\lambda_x + \varepsilon$, thereby raising social welfare. This holds true for either decreasing or increasing marginal benefits. In words, if the monopoly chooses some $\lambda_x > 0$ at some k_x, the social planner will make a strictly larger effort.

(iii) WTS: $\mathbf{W}^S(\mathbf{k}_x) > \mathbf{W}^D(\omega)\ \forall \mathbf{k}_x = \max\{\mathbf{k}_x, \mathbf{k}_y\}$. The argument follows the same line in the comparison between social planning and monopoly. Observe that only the front-runner makes positive profit flows. Thus we can collapse the duopoly state to be unidimensional and compare the social planner with the duopoly. First, there exists time T beyond which no firms innovate, allowing us to truncate the far future and compare finite sums of social welfare flows. Second, Lemma 2.3.5 already established that $\Phi^{GS}(k_x) \geq \Phi^{GD}(k_x, k_y)\ \forall k_x \geq k_y$. Third, by construction, the social planner can always follow the evolution path of the duopoly in equilibrium with equal or less R&D costs and weakly higher probabilities of success. Explicitly, let $\lambda_{\max}(k_x, k_y) = \max\{\lambda(k_x, k_y), \lambda(k_y, k_x)\}$. Given k_x, the social planner chooses $\lambda_{\max}(k_x, k_y)$, and, in the next period, will produce at the duopoly quality level. Thus, the social planner can follow this policy rule, which is weakly suboptimal, and still generates higher social welfare than under duopoly.

A.7 Nonlinear Substitution: Behavior and Welfare

Consumption Behavior. Consumers choose a budget share b given that b is optimally distributed on the innovation products. First, given some b and prices, consumers solve the problem

$$\max_{x,y\geq 0} \left\{(\theta_x x)^\alpha + (\theta_y y)^\alpha\right\} \text{ s.t. } p_x x + p_y y = b,$$

with the FOC

$$\frac{\alpha\theta_x \left(\theta_x x\right)^{\alpha-1}}{p_x} = \frac{\alpha\theta_y \left(\theta_y y\right)^{\alpha-1}}{p_y}.$$

Thus, the optimal consumption bundle is

$$x=\frac{\left(p_x/\theta_x\right)^{\alpha/(\alpha-1)}}{\left(p_x/\theta_x\right)^{\alpha/(\alpha-1)}+\left(p_y/\theta_y\right)^{\alpha/(\alpha-1)}}b=S_x\frac{b}{p_x}, \tag{A.6}$$

$$y=\frac{\left(p_y/\theta_y\right)^{\alpha/(\alpha-1)}}{\left(p_x/\theta_x\right)^{\alpha/(\alpha-1)}+\left(p_y/\theta_y\right)^{\alpha/(\alpha-1)}}b=S_y\frac{b}{p_y}. \tag{A.7}$$

Second, to solve for the optimal b, consumers maximize

$$\max_{b\in(0,B)}\left\{u\left(\theta_x^\alpha x\left(b\right)^\alpha+\theta_y^\alpha y\left(b\right)^\alpha\right)+B-b\right\}.$$

Hence

$$b\left(\theta_x,\theta_y;p_x,p_y\right)=\alpha^\pi\left[\left(\theta_xS_x/p_x\right)^\alpha+\left(\theta_yS_y/p_y\right)^\alpha\right]^{(1-\sigma)\pi}, \tag{A.8}$$

where $\pi=1/(1+\alpha\sigma-\alpha)$, and it can be shown that $\partial b/\partial\theta_x>0$, $\partial b/\partial\theta_y>0$, which hold for $\sigma\in(0,1)$. Thus given $\{\theta_x,\theta_y,p_x,p_y\}$, where prices depend on the allocation mechanism, we can calculate one-period utility.

Monopoly. The monopoly solves $\max_{p_x,p_y\geq w}\{x(p_x-w)+y(p_y-w)\}$, where the demand functions are specified earlier. Let the maximized profit function be $\Pi^M(\omega)$, which is the solution of

$$\Pi^M(\omega)=\max_{p_x,p_y\geq w}\left\{\left[S_x\left(1-\frac{w}{p_x}\right)+S_y\left(1-\frac{w}{p_y}\right)\right]b\right\}.$$

Appendix A.8 will provide more details on this pricing problem. Next, to find the optimal R&D investments, the monopoly solves the Bellman equation

$$V^M\left(\omega\right)=\max_{\lambda_x,\lambda_y\geq0}\left\{\Pi^M(\omega)-c\left(\lambda_x\right)-c\left(\lambda_y\right)+\beta E_{(\lambda_x,\lambda_y)}V^M\left(\omega'\right)\right\}.$$

Based on monopoly pricing, the one-period utility function $U^M\left(\omega\right)$ can be constructed. Conditional on the state ω, the flow of social welfare is

$$\Phi^M(\omega)=U^M\left(\omega\right)+\Pi^M(\omega)-c\left(\lambda_x\right)-c\left(\lambda_y\right). \tag{A.9}$$

Given agents' maximizing behaviors, discounted life-time social welfare is defined recursively as

$$W^M\left(\omega\right)=\Phi^M(\omega)+\beta E_{(\lambda_x,\lambda_y)}W^M\left(\omega'\right), \tag{A.10}$$

based on which we can find the maximal and *ex ante* social welfare values.

Duopoly. In every period, the firms engage in a Bertrand game. Given Y' price, X chooses a price to maximize its profit

$$\max_{p_x \geq w} \left\{ S_x b \left(1 - \frac{w}{p_x} \right) \right\}.$$

Firm Y solves a similar problem. The FOCs of these problems constitute a system which pins down the equilibrium price, conditional on the quality levels (Appendix A.8). The pricing game has an equilibrium (Caplin and Nalebuff, 1991). Observe that the maximized profit function is symmetric with respect to X and Y. Thus if $\Pi^D(\omega)$ is the profit function for X, then $\Pi^D(\widetilde{\omega})$ is the profit function for Y, where $\widetilde{\omega}$ is the permuted ω. Given the one-period profit function, firm X solves the Bellman equation

$$V^D(\omega) = \max_{\lambda_x \geq 0} \left\{ \Pi^D(\omega) - c(\lambda_x) + \beta E_{(\lambda_x, \lambda_y)} V^D(\omega') \right\},$$

knowing that firm Y also solves a similar problem. Conditional on ω, the FOCs of these two Bellman equations constitute a system which pins down the equilibrium play in that state. Again, we can solve the entire R&D game by backward induction, knowing that any firm will not invest further if already on and beyond k^*. The solution of the problem is a symmetric MPE $\lambda(\cdot)$, which specifies how much a firm will spend on R&D in a given state.

Given the pricing behavior of the duopoly conditional on ω, we can construct the one-period utility function $U^D(\omega)$. In combination, the flow of social welfare is

$$\Phi^D(\omega) = U^D(\omega) + \Pi^D(\omega) + \Pi^D(\widetilde{\omega}) - c(\lambda_x) - c(\lambda_y), \tag{A.11}$$

where $\lambda_x = \lambda(\omega)$ and $\lambda_y = \lambda(\widetilde{\omega})$. Recursively, the discounted life-time social welfare function is defined as

$$W^D(\omega) = \Phi^D(\omega) + \beta E_{(\lambda_x, \lambda_y)} W^D(\omega'). \tag{A.12}$$

Based on this equation, maximal long-run and *ex ante* social welfare values can be constructed.

The Social Planner. The social planner is different from the monopoly in two main aspects: (i) prices are set at unit cost; and (ii) investments are for consumer utility rather than profit. As prices are set at unit cost w, the one-period utility $U^S(\omega)$ can be calculated according to (A.6)-(A.9). Thus the Bellman equation is

$$V^S(\omega) = \max_{\lambda_x, \lambda_y \geq 0} \left\{ U^S(\omega) - c(\lambda_x) - c(\lambda_y) + \beta E_{(\lambda_x, \lambda_y)} V^S(\omega') \right\}. \tag{A.13}$$

Observe that the value function is exactly the discounted life-time social welfare $W^S(\omega)$. Based on this value function, we can specify the maximal and *ex ante* social welfare levels.

A.8 Memoranda: Monopoly and Duopoly Pricing

Monopoly. Here are more details about the monopoly pricing problem. Numerical solutions to the pricing problem rely on the Newton's method with analytical first-order and second-order derivatives of the objective function. Define

$$\mu_x = \frac{q_x^{\epsilon-1}}{q_x^\epsilon + q_y^\epsilon},$$

$$\mu_y = \frac{{q_y}^{\epsilon-1}}{q_x^\epsilon + q_y^\epsilon},$$

$$b = \alpha^\pi \left(\mu_x^\alpha + \mu_y^\alpha\right)^{(1-\sigma)\pi},$$

$$f = \left(1 - \frac{w}{\theta_x}\mu_x - \frac{w}{\theta_y}\mu_y\right) b,$$

where $\epsilon = \alpha/(\alpha-1)$, $q_x = p_x/\theta_x$, $q_y = p_y/\theta_y$, and f is the objective function. The first-order derivatives are

$$\frac{\partial f}{\partial p_x} = -\left(\frac{w}{\theta_x}\mu_x'(p_x) + \frac{w}{\theta_y}\mu_y'(p_x)\right) b + \left(1 - \frac{w}{\theta_x}\mu_x - \frac{w}{\theta_y}\mu_y\right) b'(p_x), \tag{A.14}$$

$$\frac{\partial f}{\partial p_y} = -\left(\frac{w}{\theta_x}\mu'_x\left(p_y\right) + \frac{w}{\theta_y}\mu'_y\left(p_y\right)\right)b + \left(1 - \frac{w}{\theta_x}\mu_x - \frac{w}{\theta_y}\mu_y\right)b'\left(p_y\right), \quad \text{(A.15)}$$

where

$$\mu'_x\left(p_x\right) = \frac{1}{\theta_x}\left[\left(\epsilon - 1\right)\frac{\mu_x}{q_x} - \epsilon\mu_x^2\right],$$

$$\mu'_x\left(p_y\right) = -\frac{\epsilon\mu_x\mu_y}{\theta_y},$$

$$\mu'_y\left(p_y\right) = \frac{1}{\theta_y}\left[\left(\epsilon - 1\right)\frac{\mu_y}{q_y} - \epsilon\mu_y^2\right],$$

$$\mu'_y\left(p_x\right) = -\frac{\epsilon\mu_x\mu_y}{\theta_x},$$

$$b'\left(p_x\right) = \frac{\alpha b}{\left(\mu_x^\alpha + \mu_y^\alpha\right)}\left[\mu_x^{(\alpha-1)}\mu'_x\left(p_x\right) + \mu_y^{(\alpha-1)}\mu'_y\left(p_x\right)\right],$$

$$b'\left(p_y\right) = \frac{\alpha b}{\left(\mu_x^\alpha + \mu_y^\alpha\right)}\left[\mu_x^{(\alpha-1)}\mu'_x\left(p_y\right) + \mu_y^{(\alpha-1)}\mu'_y\left(p_y\right)\right].$$

The second-order derivatives are

$$\begin{aligned}\frac{\partial^2 f}{\partial p_x^2} &= -\left(\frac{w}{\theta_x}\mu''_x\left(p_x, p_x\right) + \frac{w}{\theta_y}\mu''_y\left(p_x, p_x\right)\right)b \\ &\quad -2\left(\frac{w}{\theta_x}\mu'_x\left(p_x\right) + \frac{w}{\theta_y}\mu'_y\left(p_x\right)\right)b'\left(p_x\right) \\ &\quad +\left(1 - \frac{w}{\theta_x}\mu_x - \frac{w}{\theta_y}\mu_y\right)b''\left(p_x, p_x\right), \qquad \text{(A.16)}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial p_y^2} &= -\left(\frac{w}{\theta_x}\mu''_x\left(p_y, p_y\right) + \frac{w}{\theta_y}\mu''_y\left(p_y, p_y\right)\right)b \\ &\quad -2\left(\frac{w}{\theta_x}\mu'_x\left(p_y\right) + \frac{w}{\theta_y}\mu'_y\left(p_y\right)\right)b'\left(p_y\right) \\ &\quad +\left(1 - \frac{w}{\theta_x}\mu_x - \frac{w}{\theta_y}\mu_y\right)b''\left(p_y, p_y\right), \qquad \text{(A.17)}\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial p_x p_y} &= -\left(\frac{w}{\theta_x}\mu_x''\left(p_x,p_y\right)+\frac{w}{\theta_y}\mu_y''\left(p_x,p_y\right)\right)b \\
&\quad -\left(\frac{w}{\theta_x}\mu_x'\left(p_x\right)+\frac{w}{\theta_y}\mu_y'\left(p_x\right)\right)b'\left(p_y\right) \\
&\quad -\left(\frac{w}{\theta_x}\mu_x'\left(p_y\right)+\frac{w}{\theta_y}\mu_y'\left(p_y\right)\right)b'\left(p_x\right) \\
&\quad +\left(1-\frac{w}{\theta_x}\mu_x-\frac{w}{\theta_y}\mu_y\right)b''\left(p_x,p_y\right), \qquad \text{(A.18)}
\end{aligned}
$$

where

$$\mu_x''\left(p_x,p_x\right)=\frac{1}{\theta_x}\left[\left(\epsilon-1\right)\frac{\mu_x'(p_x)q_x-\mu_x/\theta_x}{q_x^2}-2\epsilon\mu_x\mu_x'\left(p_x\right)\right],$$

$$\mu_x''\left(p_y,p_y\right)=-\frac{\epsilon}{\theta_y}\left[\mu_x'\left(p_y\right)\mu_y+\mu_x\mu_y'\left(p_y\right)\right],$$

$$\mu_x''\left(p_x,p_y\right)=-\frac{\epsilon}{\theta_y}\left[\mu_x'\left(p_x\right)\mu_y+\mu_x\mu_y'\left(p_x\right)\right],$$

$$\mu_y''\left(p_y,p_y\right)=\frac{1}{\theta_y}\left[\left(\epsilon-1\right)\frac{\mu_y'\left(p_y\right)q_y-\mu_y/\theta_y}{q_y^2}-2\epsilon\mu_y\mu_y'\left(p_y\right)\right],$$

$$\mu_y''\left(p_x,p_x\right)=-\frac{\epsilon}{\theta_x}\left[\mu_x'\left(p_x\right)\mu_y+\mu_x\mu_y'\left(p_x\right)\right],$$

$$\mu_y''\left(p_x,p_y\right)=-\frac{\epsilon}{\theta_x}\left[\mu_x'\left(p_y\right)\mu_y+\mu_x\mu_y'\left(p_y\right)\right],$$

$$b''\left(p_x,p_x\right)=\frac{\alpha b}{\left(\mu_x^\alpha+\mu_y^\alpha\right)}\left[\begin{array}{c}\left(\alpha-1\right)\mu_x^{(\alpha-2)}\left(\mu_x'(p_x)\right)^2+\mu_x^{(\alpha-1)}\mu_x''\left(p_x,p_x\right)\\ +\left(\alpha-1\right)\mu_y^{(\alpha-2)}\left(\mu_y'\left(p_x\right)\right)^2+\mu_y^{(\alpha-1)}\mu_y''\left(p_x,p_x\right)\end{array}\right],$$

$$b''\left(p_y,p_y\right)=\frac{\alpha b}{\left(\mu_x^\alpha+\mu_y^\alpha\right)}\left[\begin{array}{c}\left(\alpha-1\right)\mu_x^{(\alpha-2)}\left(\mu_x'\left(p_y\right)\right)^2+\mu_x^{(\alpha-1)}\mu_x''\left(p_y,p_y\right)\\ +\left(\alpha-1\right)\mu_y^{(\alpha-2)}\left(\mu_y'\left(p_y\right)\right)^2+\mu_y^{(\alpha-1)}\mu_y''\left(p_y,p_y\right)\end{array}\right],$$

$$
\begin{aligned}
b''\left(p_x,p_y\right) &= \frac{\alpha\left\{b'\left(p_y\right)\left(\mu_x^\alpha+\mu_y^\alpha\right)-\alpha b\Sigma_y\right\}}{\left(\mu_x^\alpha+\mu_y^\alpha\right)^2}\Sigma_x \\
&+\frac{\alpha b}{\left(\mu_x^\alpha+\mu_y^\alpha\right)}\left[\left(\alpha-1\right)\mu_x^{(\alpha-2)}\mu_x'\left(p_x\right)\mu_x'\left(p_y\right)+\mu_x^{(\alpha-1)}\mu_x''\left(p_x,p_y\right)\right] \\
&+\frac{\alpha b}{\left(\mu_x^\alpha+\mu_y^\alpha\right)}\left[\left(\alpha-1\right)\mu_y^{(\alpha-2)}\mu_y'\left(p_x\right)\mu_y'\left(p_y\right)+\mu_y^{(\alpha-1)}\mu_y''\left(p_x,p_y\right)\right],
\end{aligned}
$$

with

$$\Sigma_x = \left[\mu_x^{(\alpha-1)}\mu_x'(p_x) + \mu_y^{(\alpha-1)}\mu_y'(p_x)\right],$$

$$\Sigma_y = \left[\mu_x^{(\alpha-1)}\mu_x'(p_y) + \mu_y^{(\alpha-1)}\mu_y'(p_y)\right].$$

Duopoly. Numerical solutions to the Nash pricing problem also rely on the Newton's method with the analytical Jacobian of the system which is built from firms' first-order conditions with respect to prices. Let $f(p_x, p_y) = (f_x\ f_y)'$ where $\{f_x, f_y\}$ are the first-order derivatives of X's and Y's profit functions $\{\theta_x\Pi_X, \theta_y\Pi_Y\}$, respectively. Thus, a Nash equilibrium satisfies $f(p_x, p_y) = 0$. We have

$$\theta_x\Pi_X(p_x, p_y) = \mu_x b(p_x - w),$$

$$\theta_y\Pi_Y(p_x, p_y) = \mu_y b(p_y - w),$$

$$f_x = (p_x - w)\left[\mu_x'(p_x)b + \mu_x b'(p_x)\right] + \mu_x b, \tag{A.19}$$

$$f_y = (p_y - w)\left[\mu_y'(p_y)b + \mu_y b'(p_y)\right] + \mu_y b. \tag{A.20}$$

The associated Jacobian is

$$J = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}, \tag{A.21}$$

where

$$\begin{aligned} f_{xx} &= 2\left[\mu_x'(p_x)b + \mu_x b'(p_x)\right] + \\ &\quad (p_x - w)\left[\mu_x''(p_x, p_x)b + 2\mu_x'(p_x)b'(p_x) + \mu_x b''(p_x, p_x)\right], \end{aligned}$$

$$\begin{aligned} f_{yy} &= 2\left[\mu_y'(p_y)b + \mu_y b'(p_y)\right] + \\ &\quad (p_y - w)\left[\mu_y''(p_y, p_y)b + 2\mu_y'(p_y)b'(p_y) + \mu_y b''(p_y, p_y)\right], \end{aligned}$$

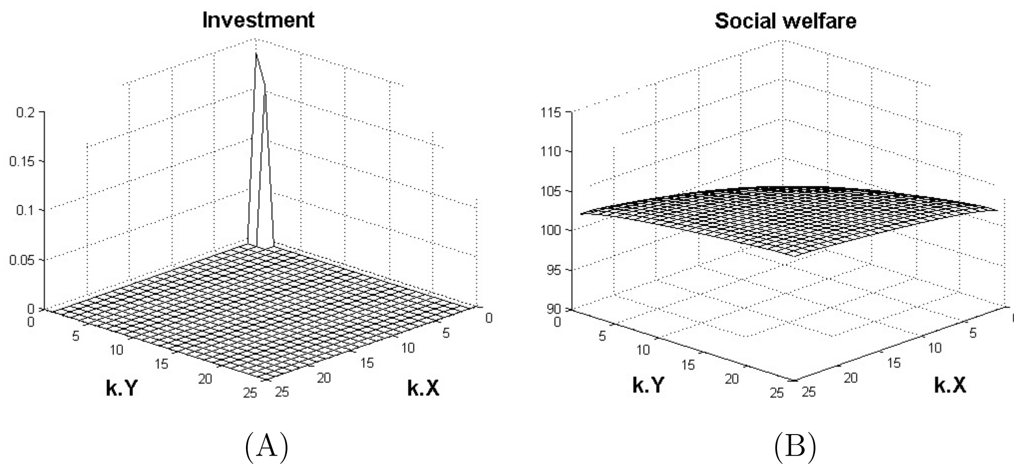
$$f_{xy} = \left[\mu_x'(p_y)b + \mu_x b'(p_y)\right] +$$
$$(p_x - w)\left[\mu_x''(p_x, p_y)b + \mu_x'(p_x)b'(p_y) + \mu_x'(p_y)b'(p_x) + \mu_x b''(p_x, p_y)\right],$$

$$f_{yx} = \left[\mu_y'(p_x)\, b + \mu_y b'(p_x)\right]$$
$$(p_y - w)\left[\mu_y''(p_y, p_x)\, b + \mu_y'(p_y)\, b'(p_x) + \mu_y'(p_x)\, b'(p_y) + \mu_y b''(p_y, p_x)\right],$$

where all of the denotations are from the monopoly pricing problem.

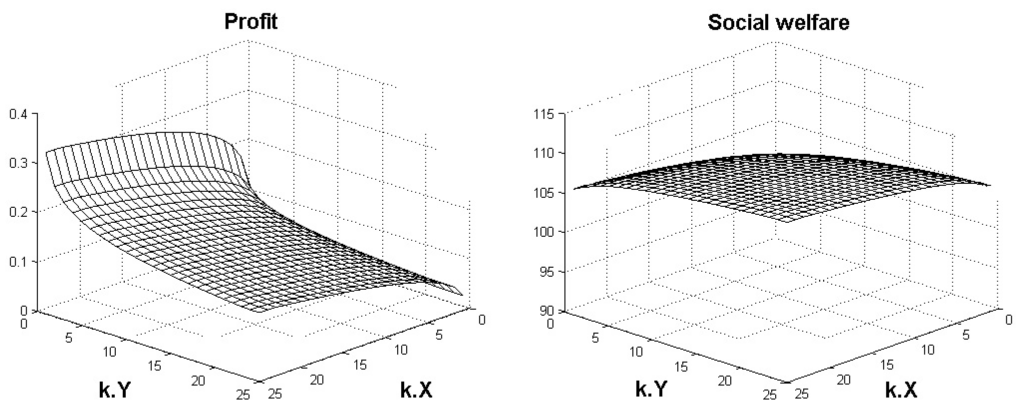
A.9 Nonlinear Substitution: Characterizations

Figure A.1: Monopoly: investment & social welfare



Note: $\alpha = 0.8$. In panel A, the monopoly only innovates product x at low know-how stocks. In panel B, the social welfare function is increasing and concave. Given the monopoly's innovation behavior, the economy only progresses for a short period of time and achieves a maximal welfare level associated with low product quality.

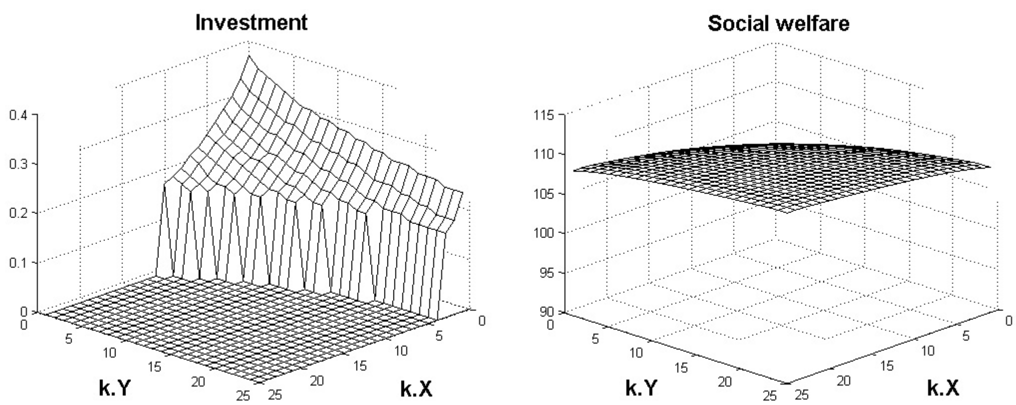
Figure A.2: Duopoly: profit & social welfare



(A) (B)

Note: $\alpha = 0.8$. In panel A, firm X's one-period profit is increasing in its product quality and decreasing in that of the rival; Hence, X's innovation incentive is greater when θ_y is smaller. For low θ_y, firm X innovates its product all the way to k^* and faces bouncing effects at this threshold; For this reason, R&D efforts along this belt look bumpy; If k^* is large enough, beyond which the slope of θ is infinitesimal, the bumpy effects disappear. In panel B, the welfare function is also increasing and concave.

Figure A.3: Social planner: investment & social welfare



(A) (B)

Note: $\alpha = 0.8$. In panel A, innovation investment in x is decreasing in both k_x and k_y. In panel B, the planner social welfare function has the same shape but higher value than those in the other two market structures.

Appendix B

Derivations for Chapter 3

B.1 Equilibrium in the Basic Model

(i) Agent i solves the UMP in period t

$$\max_{\{a_{it},b_{it}\}} \left\{ (\alpha_t a_{it})^\theta + (\beta_t b_{it})^\theta \right\}^{1/\theta}$$
$$\text{s.t. } a_{it} + p_t b_{it} = e_{ait} + p_t e_{bit}.$$

Equivalently, given the Inada condition, we need to solve

$$\max_{b_{it}>0} \left\{ \alpha_t^\theta \left(e_{ait} + p_t e_{bit} - p_t b_{it}\right)^\theta + \beta_t^\theta b_{it}^\theta \right\}^{1/\theta}.$$

The necessary and sufficient condition with respect to b_{it} is

$$\alpha_t^\theta \left(e_{ait} + p_t e_{bit} - p_t b_{it}\right)^{\theta-1} p_t = \beta_t^\theta b_{it}^{\theta-1}. \tag{B.1}$$

(ii) In equilibrium, we already have that $b_{it} = B_t$. In addition, with the equal-endowment rule, $e_{ait} = A_t$ and $e_{bit} = B_t$. Thus (B.1) can be written as

$$\alpha_t^\theta A_t^{\theta-1} p_t = \beta_t^\theta B_t^{\theta-1},$$

and the equilibrium relative price is

$$p_t = \left(\frac{\beta_t}{\alpha_t}\right)^\theta \left(\frac{B_t}{A_t}\right)^{\theta-1}. \tag{B.2}$$

(iii) Given the equilibrium relative price in (B.2), the equilibrium budget share for commodity a is

$$S_{at} = \frac{1}{1 + p_t \frac{B_t}{A_t}}, \text{ or}$$
$$S_{at} = \frac{1}{1 + \left(\frac{\beta_t}{\alpha_t}\right)^\theta \left(\frac{B_t}{A_t}\right)^\theta}. \tag{B.3}$$

Figure B.1: Quality innovation and marginal utility, $\theta < 0$

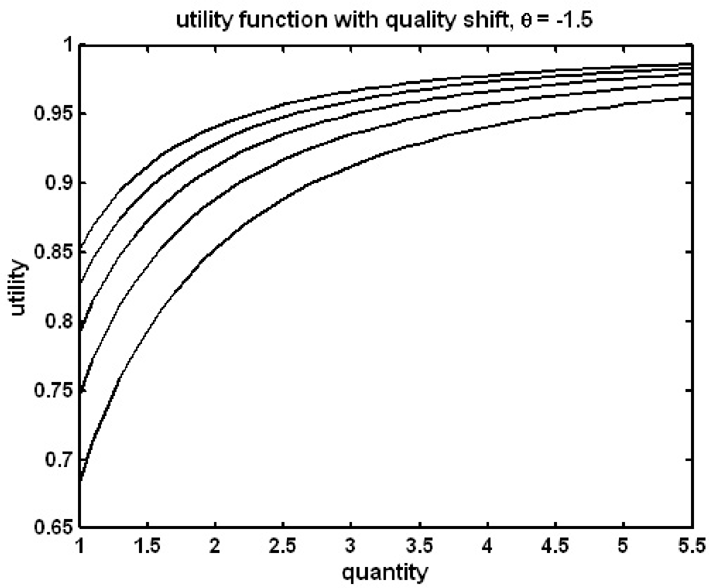
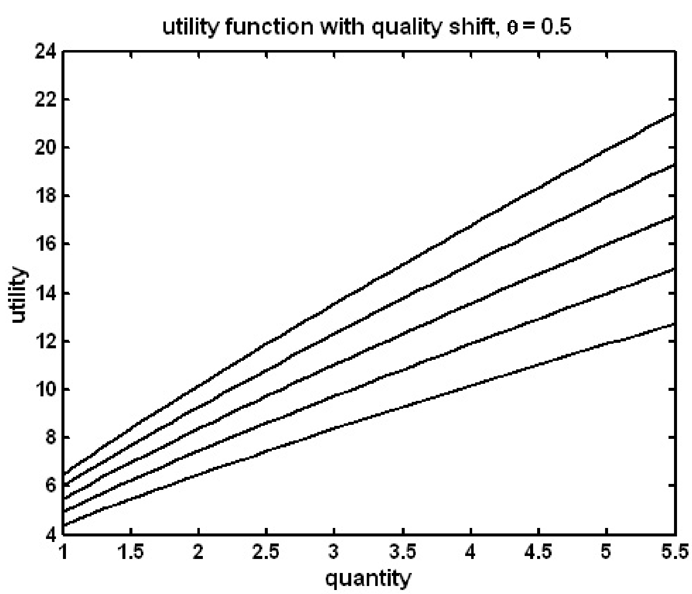


Figure B.2: Quality innovation and marginal utility, $\theta > 0$



B.2 Relations between VAR Parameters

(i) Derivation of variances of $(\varepsilon_{pt}, \varepsilon_{qt})$

From (3.14) and (3.15), we have

$$\begin{cases} P_t = \lambda_{pp} P_{t-1} + \lambda_{qp} Q_{t-1} + \varepsilon_{pt} \\ Q_t = \lambda_{pq} P_{t-1} + \lambda_{qq} Q_{t-1} + \varepsilon_{qt} \end{cases} \tag{B.4}$$

$$\Longrightarrow \begin{cases} var(P_t) = var(\lambda_{pp} P_{t-1} + \lambda_{qp} Q_{t-1} + \varepsilon_{pt}) \\ var(Q_t) = var(\lambda_{pq} P_{t-1} + \lambda_{qq} Q_{t-1} + \varepsilon_{qt}) \end{cases}$$

$$\Longrightarrow \begin{cases} \sigma_p^2 = \lambda_{pp}^2 \sigma_p^2 + \lambda_{qp}^2 \sigma_q^2 + \gamma_p^2 + 2\lambda_{pp}\lambda_{qp}\sigma_{pq} \\ \sigma_q^2 = \lambda_{pq}^2 \sigma_p^2 + \lambda_{qq}^2 \sigma_q^2 + \gamma_q^2 + 2\lambda_{pq}\lambda_{qq}\sigma_{pq} \end{cases}$$

$$\Longrightarrow \begin{bmatrix} \gamma_p^2 \\ \gamma_q^2 \end{bmatrix} = \begin{bmatrix} 1-\lambda_{pp}^2 & -\lambda_{qp}^2 \\ -\lambda_{pq}^2 & 1-\lambda_{qq}^2 \end{bmatrix} \begin{bmatrix} \sigma_p^2 \\ \sigma_q^2 \end{bmatrix} - \begin{bmatrix} 2\lambda_{pp}\lambda_{qp}\sigma_{pq} \\ 2\lambda_{pq}\lambda_{qq}\sigma_{pq} \end{bmatrix}. \tag{B.5}$$

(ii) Derivation of covariance of $(\varepsilon_{pt}, \varepsilon_{qt})$

From (B.4) and the fact that $E(P_t) = E(Q_t) = 0$, we have

$$\begin{aligned} covar\,(P_t, Q_t) &= \sigma_{pq} = E(P_t Q_t) \\ &= E\left[(\lambda_{pp} P_{t-1} + \lambda_{qp} Q_{t-1} + \varepsilon_p)(\lambda_{pq} P_{t-1} + \lambda_{qq} Q_{t-1} + \varepsilon_q)\right] \\ &= \lambda_{pp}\lambda_{pq}\sigma_p^2 + \lambda_{qp}\lambda_{qq}\sigma_q^2 + (\lambda_{pp}\lambda_{qq} + \lambda_{pq}\lambda_{qp})\,\sigma_{pq} + \gamma_{pq}. \end{aligned}$$

Thus

$$\gamma_{pq} = \left[1 - (\lambda_{pp}\lambda_{qq} + \lambda_{pq}\lambda_{qp})\right]\sigma_{pq} - \left(\lambda_{pp}\lambda_{pq}\sigma_p^2 + \lambda_{qp}\lambda_{qq}\sigma_q^2\right). \tag{B.6}$$

B.3 Estimation of Relative Quality Index

(i) Deriving relative quality index:

Let $[(\beta/\alpha)_t\, u]_{est}^{\theta}$ be the modified relative quality index to be estimated in period t. The index is chosen as follows

$$\left[\left(\frac{\beta}{\alpha}\right)_t u\right]_{est}^{\theta} = \arg\min_x \left\{U_t' W U_t\right\} \tag{B.7}$$

where U_t and W are defined in the main text. Let the objective function be

$$F(x) = U_t^{'} W U_t$$
$$F(x) = \frac{1}{Det(\Omega)} [\widetilde{u}_{1t}\ \widetilde{u}_{2t}] \begin{bmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{bmatrix} \begin{bmatrix} \widetilde{u}_{1t} \\ \widetilde{u}_{2t} \end{bmatrix}$$
$$F(x) = \frac{\sigma_2^2 (C_{1t}x - 1)^2 + \sigma_1^2 (C_{2t}x - 1)^2 - 2\sigma_{12} (C_{1t}x - 1)(C_{2t}x - 1)}{Det(\Omega)}.$$

The FOC and also SOC is

$$F^{'}(x) = 0$$
$$\Longrightarrow \left[\sigma_2^2 C_{1t}^2 + \sigma_1^2 C_{2t}^2 - 2\sigma_{12} C_{1t} C_{2t}\right] x = \left[\sigma_2^2 C_{1t} + \sigma_1^2 C_{2t} - \sigma_{12} (C_{1t} + C_{2t})\right]$$

Finally, the estimated modified index is

$$x = \frac{\sigma_2^2 C_{1t} + \sigma_1^2 C_{2t} - \sigma_{12} (C_{1t} + C_{2t})}{\sigma_2^2 C_{1t}^2 + \sigma_1^2 C_{2t}^2 - 2\sigma_{12} C_{1t} C_{2t}}. \tag{B.8}$$

(ii) The two-step procedure

In the first step, the variance-covariance matrix is

$$\Omega_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the corresponding solution is

$$x = \frac{C_{1t} + C_{2t}}{C_{1t}^2 + C_{2t}^2}. \tag{B.9}$$

The estimated errors are

$$\widehat{\widetilde{u}}_{1t} = \frac{C_{1t}C_{2t} - C_{2t}^2}{C_{1t}^2 + C_{2t}^2}, \tag{B.10}$$
$$\widehat{\widetilde{u}}_{2t} = \frac{C_{1t}C_{2t} - C_{1t}^2}{C_{1t}^2 + C_{2t}^2}. \tag{B.11}$$

Let $e_{1t} = \widehat{\widetilde{u}}_{1t} - \left(\sum \widehat{\widetilde{u}}_{1t}\right)/T$ and $e_{2t} = \widehat{\widetilde{u}}_{2t} - \left(\sum \widehat{\widetilde{u}}_{2t}\right)/T$, we have $\widehat{\Omega} = E^{'}E$, where

$$E = \begin{bmatrix} e_{11} & e_{21} \\ .. & .. \\ e_{1t} & e_{2t} \\ .. & .. \\ e_{1T} & e_{2T} \end{bmatrix},$$

and use $\widehat{\Omega}$ for the second step estimation.

(iii) Conditional expectation and variance of estimated quality index

Let q_t be a true quality index (up to some unknown scale) and $(\widetilde{u}_{1t}, \widetilde{u}_{2t})$ be defined in (3.27), the estimated quality index based on (B.8) can be rewritten as

$$\Phi\left(\widetilde{u}_{1t}, \widetilde{u}_{2t}; q_t\right) = q_t \left[\frac{\Phi_{Ut}}{\Phi_{Lt}}\right]^{1/\theta} \tag{B.12}$$

where

$$\begin{aligned}
\Phi_{Ut} &= \sigma_2^2\left(\widetilde{u}_{1t}+1\right) + \sigma_1^2\left(\widetilde{u}_{2t}+1\right) - \sigma_{12}\left(\widetilde{u}_{1t}+\widetilde{u}_{2t}+2\right), \\
\Phi_{Lt} &= \sigma_2^2\left(\widetilde{u}_{1t}+1\right)^2 + \sigma_1^2\left(\widetilde{u}_{2t}+1\right)^2 - 2\sigma_{12}\left(\widetilde{u}_{1t}+1\right)\left(\widetilde{u}_{2t}+1\right).
\end{aligned}$$

The estimated index has the following conditional expectation

$$E\left[\Phi\left(\widetilde{u}_{1t}, \widetilde{u}_{2t}; q_t\right)|q_t\right] = q_t E\left\{\left[\frac{\Phi_{Ut}}{\Phi_{Lt}}\right]^{1/\theta}\right\}. \tag{B.13}$$

Let $\widetilde{\mu}_1 = E\left(\widetilde{u}_{1t}\right)$ and $\widetilde{\mu}_2 = E\left(\widetilde{u}_{2t}\right)$. By the Delta method with reference to the means, conditional variance of the estimated quality index is

$$var\left(\Phi|q_t\right) = \Delta\Phi\left(\widetilde{\mu}_1, \widetilde{\mu}_2; q_t\right)' \Omega \Delta\Phi\left(\widetilde{\mu}_1, \widetilde{\mu}_2; q_t\right) \tag{B.14}$$

where

$$\Delta\Phi\left(\widetilde{\mu}_1, \widetilde{\mu}_2; q_t\right) = \left[\begin{array}{c} \partial\Phi\left(\widetilde{u}_{1t}, \widetilde{u}_{2t}; q_t\right)/\partial\widetilde{u}_{1t} \\ \partial\Phi\left(\widetilde{u}_{1t}, \widetilde{u}_{2t}; q_t\right)/\partial\widetilde{u}_{2t} \end{array}\right]_{\left(\widetilde{\mu}_1, \widetilde{\mu}_2\right)},$$

specifically

$$\begin{aligned}
\frac{\partial\Phi}{\partial\widetilde{u}_{1t}} &= D_t\left[\frac{\left(\sigma_2^2-\sigma_{12}\right)\Phi_{Lt} - 2\left[\sigma_2^2\left(\widetilde{u}_{1t}+1\right) - \sigma_{12}\left(\widetilde{u}_{2t}+1\right)\right]\Phi_{Ut}}{\Phi_{Lt}^2}\right], \\
\frac{\partial\Phi}{\partial\widetilde{u}_{2t}} &= D_t\left[\frac{\left(\sigma_1^2-\sigma_{12}\right)\Phi_{Lt} - 2\left[\sigma_1^2\left(\widetilde{u}_{2t}+1\right) - \sigma_{12}\left(\widetilde{u}_{1t}+1\right)\right]\Phi_{Ut}}{\Phi_{Lt}^2}\right], \\
D_t &= \frac{q_t}{\theta}\left[\frac{\Phi_{Ut}}{\Phi_{Lt}}\right]^{1/\theta-1}.
\end{aligned}$$

In empirical studies, the correction factor $E\left[(\Phi_{Ut}/\Phi_{Lt})^{1/\theta}\right]$ in (B.13) is estimated by

$$E\left\{\left[\frac{\Phi_{Ut}}{\Phi_{Lt}}\right]^{1/\theta}\right\}_{est} = \frac{1}{T}\sum\nolimits_{t=1}^{T}\left[\frac{\Phi_{Ut}}{\Phi_{Lt}}\right]^{1/\widehat{\theta}}, \tag{B.15}$$

where $(\widetilde{u}_{1t}, \widetilde{u}_{2t})$ are derived from the multiplicative residuals in the second-step estimation. If the estimated correction factor is significantly different from unit, the point and variance estimates of the estimated relative quality index should be adjusted accordingly.

B.4 US Services-Goods Data Set

The annual data set on US services and goods covers the period 1946-2006. The series are mainly retrieved from NIPA tables which are reported by the Bureau of Economic Analysis. The series on population is from the estimates of the US Census Bureau. Classifications of goods and services follow the definitions of NIPA tables. The broad components of goods industries are agriculture, forestry, and fisheries; mining; and manufacturing. The services industries are transportation and public utilities; wholesale trade; retail trade and automobile services; finance, insurance, and real estate; different services; and government services.

The original data set has the following variables: (1) US population index (US Census); (2) goods quantity index (NIPA 1.2.3); (3) goods price index (NIPA 1.2.4); (4) services quantity index (NIPA 1.2.3); (5) services price index (NIPA 1.2.4); and (6) budget share for goods (NIPA 1.5.5). It is noted that we leave residential and non-residential structures out of the data set. Some data features are worth noted as follows.

First, goods are both durable and nondurable. we rely on quantity flows of new durable goods rather than service flows from durable stocks. The reason for not using services flows is that stocks of durable goods are composed of different quality levels which are unknown. The same reason applies to the omission of residential and non-residential structures, which can render services for a very long period of time.

Second, the bottom line of the current NIPA tables is that: "...Percent changes in real GDP and its components are equal to the percent changes of the quantity indexes; percent changes in prices are equal to the percent changes of the price indexes..." (A Guide to the NIPA's by the BEA, 2001). Technically, chain-type quantity and price indices are based on Fisher (F) formula which uses weights from two adjacent years, i.e. a combination of Laspeyres (L) and Paasche (P) indices. Specifically, let q's and p's be quantities and prices, Fisher quantity index of period t relative to that of period $t-1$ is

$$Q_t^F = \sqrt{Q_t^L \times Q_t^P}, \tag{B.16}$$

where

$$\begin{aligned} Q_t^L &= \frac{\sum p_{t-1} q_t}{\sum p_{t-1} q_{t-1}} \\ Q_t^P &= \frac{\sum p_t q_t}{\sum p_t q_{t-1}}; \end{aligned}$$

and by the same token, Fisher price index of period t is

$$P_t^F = \sqrt{P_t^L \times P_t^P}, \tag{B.17}$$

where

$$\begin{aligned} P_t^L &= \frac{\sum p_t q_{t-1}}{\sum p_{t-1} q_{t-1}} \\ P_t^P &= \frac{\sum p_t q_t}{\sum p_{t-1} q_t}. \end{aligned}$$

Correspondingly, the value index is defined as

$$V_t = P_t^F . Q_t^F \tag{B.18}$$

The intuition behind (B.16) and (B.17) is that if quantities or prices do not change, $Q_t^F = 1$ or $P_t^F = 1$, respectively. To put it differently, (B.16) reflects only changes in aggregate quantity, and (B.17) is only for variations in aggregate price. The product $Q_t^F \times P_t^F$ is the growth rate of the nominal value between time t and time $t-1$ (B.18). Based on this observation, in practice, most GDP components' nominal values and price indices are derived first from different Federal Government surveys. Then, starting with the most detailed level for which all the necessary data are available, nominal values are deflated to have real values or quantities (NIPA Help, BEA website).

Third, the construction of year-to-year quantity and price indices are based on the set of commodities existing in two adjacent years. If the set of varieties not shared between two adjacent years is relatively small, which is highly likely the case, time series of aggregate quantity and aggregate price are reliable for the quality inference procedure.

Besides the variables which will be used in the separation exercise, we look at the composition of GDP from the expenditure perspective to see how much the United States depends on the Rest of the World. Table B.1 shows that we can treat the US as relatively closed.

Table B.1: Relative completion of the US economy 1946-2006, percent

Accounts	1946	2006	1946-2006
Gross domestic product	**100.0**	**100.0**	**100.0**
Personal consumption expenditures	**64.9**	**70.0**	**64.5**
Goods	44.3	28.6	33.8
Services	20.6	41.4	30.7
Gross private domestic investment	**14.0**	**16.7**	**16.0**
Goods	7.2	7.8	7.6
Structures	6.8	8.9	8.4
Net exports of goods and services	**3.2**	**−5.7**	**−0.7**
Exports	6.4	11.1	7.5
Goods	5.3	7.8	5.6
Services	1.1	3.3	1.9
Imports	3.2	16.8	8.2
Goods	2.3	14.2	6.6
Services	0.9	2.6	1.6
Government expenditures & investment	**17.8**	**19.0**	**20.2**

Source: Table 1.5.5, NIPA, US Bureau of Economic Analysis.

Appendix C

Derivations for Chapter 4

C.1 Social Planner Problem: Necessary Conditions

We are solving the dynamic programming problem laid out in Definition 4.2.1. There are some notes about the choice variables in Table C.1. It is noted that the effective number of decisions is eight.

Table C.1: Notes about choice variables

description	range	notes
labor for Y_{at}	$0 < l_{at} < 1$	$l_{at} > 0$ for $\mu \in (0,1)$
labor for Y_{bt}	$0 < l_{bt} < 1$	$l_{bt} > 0$ for $\gamma \in (0,1)$
labor for α_t	$0 < n_{at} < 1$	$n_{at} > 0$ for $\xi \in (0,1)$
labor for β_t	$0 < n_{bt} < 1$	$n_{bt} > 0$ for $\xi \in (0,1)$
capital for Y_{at}	$0 < k_{at} < k_t$	$k_{at} > 0$ for $\mu \in (0,1)$
capital for Y_{bt}	$0 < k_{bt} < k_t$	$k_{bt} > 0$ for $\gamma \in (0,1)$
consumption of a	$0 < a_t < Y_{at}$	replaced, $a_t = Y_{at} - x_{at}$
consumption of b	$0 < b_t < Y_{bt}$	replaced, $b_t = Y_{bt} - x_{bt}$
investment from a	$0 < x_{at} < Y_{at}$	$x_{at} > 0$ for $\nu \in (0,1)$
investment from b	$0 < x_{bt} < Y_{bt}$	$x_{bt} > 0$ for $\nu \in (0,1)$

Recall that $\omega_t = \{\alpha_{t-1}, \beta_{t-1}, k_t, A_t, B_t\}$. Let $\{\mu_{1t}, \mu_{2t}\}$ be the Lagrangian multipliers for the capital and labor constraints in period t. The Bellman equation in (4.25) can be rewritten as

$$\begin{aligned} V(\omega_t) &= \max_{C_{1t}, C_{2t}} L(\omega_t), \qquad \text{(C.1)} \\ L(\omega_t) &= u(Y_{at} - x_{at}, Y_{bt} - x_{bt}; \alpha_t, \beta_t) + \lambda E_t V(\omega_{t+1}) \\ &\quad + \mu_{1t}(1 - l_{at} - l_{bt} - n_{at} - n_{bt}) \\ &\quad + \mu_{2t}(k_t - k_{at} - k_{bt}). \end{aligned}$$

To simplify the upcoming expressions, let

$$c_t = \left[\alpha_t^\theta \left(Y_{at} - x_{at}\right)^\theta + \beta_t^\theta \left(Y_{bt} - x_{bt}\right)^\theta\right]^{1/\theta} \tag{C.2}$$

$$d_t = \left[\alpha_t^\theta \left(Y_{at} - x_{at}\right)^\theta + \beta_t^\theta \left(Y_{bt} - x_{bt}\right)^\theta\right]^{1/\theta-1} \tag{C.3}$$

$$u_t = \frac{c_t^{1-\sigma} - 1}{1-\sigma}. \tag{C.4}$$

The necessary and also sufficient conditions are

l_{at} :

$$c_t^{-\sigma} d_t \alpha_t^\theta a_t^{\theta-1} \left(1-\mu\right) Y_{at}/l_{at} = \mu_{1t} \tag{C.5}$$

l_{bt} :

$$c_t^{-\sigma} d_t \beta_t^\theta b_t^{\theta-1} \left(1-\gamma\right) Y_{bt}/l_{bt} = \mu_{1t} \tag{C.6}$$

k_{at} :

$$c_t^{-\sigma} d_t \alpha_t^\theta a_t^{\theta-1} \mu Y_{at}/k_{at} = \mu_{2t} \tag{C.7}$$

k_{bt} :

$$c_t^{-\sigma} d_t \beta_t^\theta b_t^{\theta-1} \gamma Y_{bt}/k_{bt} = \mu_{2t} \tag{C.8}$$

n_{at} :

$$\begin{gathered} c_t^{-\sigma} d_t a_t^\theta \alpha_t^{\theta-1} \xi \left(n_{at}\right)^{\xi-1} \\ +\lambda E_t \left[c_{t+1}^{-\sigma} d_{t+1} a_{t+1}^\theta \alpha_{t+1}^{\theta-1} \left(1-\eta_\alpha\right) \xi \left(n_{at}\right)^{\xi-1}\right] = \mu_{1t} \end{gathered} \tag{C.9}$$

n_{bt} :

$$\begin{gathered} c_t^{-\sigma} d_t b_t^\theta \beta_t^{\theta-1} \xi \left(n_{bt}\right)^{\xi-1} \\ +\lambda E_t \left[c_{t+1}^{-\sigma} d_{t+1} b_{t+1}^\theta \beta_{t+1}^{\theta-1} \left(1-\eta_\beta\right) \xi \left(n_{bt}\right)^{\xi-1}\right] = \mu_{1t} \end{gathered} \tag{C.10}$$

x_{at} :

$$\begin{gathered} c_t^{-\sigma} d_t \alpha_t^\theta a_t^{\theta-1} = \\ +\lambda E_t \left[c_{t+1}^{-\sigma} d_{t+1} \alpha_{t+1}^\theta a_{t+1}^{\theta-1} \mu \left(Y_{at+1}/k_{at+1}\right) \nu \left(\phi x_{at}^{\nu-1} x_{bt}^{1-\nu}\right)\right] \end{gathered} \tag{C.11}$$

x_{bt} :

$$c_t^{-\sigma} d_t \beta_t^{\theta} b_t^{\theta-1} = \\ +\lambda E_t \left[c_{t+1}^{-\sigma} d_{t+1} \beta_{t+1}^{\theta} b_{t+1}^{\theta-1} \gamma \left(Y_{bt+1}/k_{bt+1} \right) (1-\nu) \left(\phi x_{at}^{\nu} x_{bt}^{-\nu} \right) \right] \tag{C.12}$$

μ_{1t} :

$$l_{at} + l_{bt} + n_{at} + n_{bt} = 1; \;\; \mu_{1t} > 0 \tag{C.13}$$

μ_{2t} :

$$k_{at} + k_{bt} = k_t; \;\; \mu_{2t} > 0. \tag{C.14}$$

C.2 Social Planner Problem: the Steady State

We are now solving for the nonstochastic steady state which satisfies

$$\begin{cases} \text{state:} & A_t = A, B_t = B, \alpha_t = \alpha, \beta_t = \beta, k_t = k; \\ \text{labor:} & l_{at} = l_a, l_{bt} = l_b, n_{at} = n_a, n_{bt} = n_b; \\ \text{capital:} & k_{at} = k_a, k_{bt} = k_b; \\ \text{output:} & Y_{at} = Y_a, Y_{bt} = Y_b; \\ \text{uses:} & a_t = a, b_t = b, x_{at} = x_a, x_{bt} = x_b; \\ \text{multipliers:} & \mu_{1t} = \mu_1, \mu_{2t} = \mu_2. \end{cases}$$

1) There are 13 conditions for 13 unknowns (l_a, l_b), (n_a, n_b),(k, k_a, k_b), (α, β), (x_a, x_b), (μ_1, μ_2)

$$c^{-\sigma} d\alpha^{\theta} a^{\theta-1} (1-\mu) (Y_a/l_a) = \mu_1 \tag{C.15}$$

$$c^{-\sigma} d\beta^{\theta} b^{\theta-1} (1-\gamma) (Y_b/l_b) = \mu_1 \tag{C.16}$$

$$c^{-\sigma} d\alpha^{\theta} a^{\theta-1} \mu (Y_a/k_a) = \mu_2 \tag{C.17}$$

$$c^{-\sigma} d\beta^{\theta} b^{\theta-1} \gamma (Y_b/k_b) = \mu_2 \tag{C.18}$$

$$c^{-\sigma} d a^{\theta} \alpha^{\theta-1} \left[1 + \lambda (1 - \eta_{\alpha})\right] \xi (n_a)^{\xi-1} = \mu_1 \tag{C.19}$$

$$c^{-\sigma} d b^{\theta} \beta^{\theta-1} \left[1 + \lambda (1 - \eta_{\beta})\right] \xi (n_b)^{\xi-1} = \mu_1 \tag{C.20}$$

$$1 = \lambda \mu (Y_a/k_a) \nu \left(\phi x_a^{\nu-1} x_b^{1-\nu} \right) \tag{C.21}$$

$$1 = \lambda\gamma\left(Y_b/k_b\right)(1-\nu)\left(\phi x_a^{\nu} x_b^{-\nu}\right) \tag{C.22}$$

$$n_a = \left(\eta_\alpha \alpha\right)^{1/\xi} \tag{C.23}$$

$$n_b = \left(\eta_\beta \beta\right)^{1/\xi} \tag{C.24}$$

$$\phi x_a^{\nu} x_b^{1-\nu} = \delta k \tag{C.25}$$

$$l_a + l_b = 1 - \left[\left(\eta_\alpha \alpha\right)^{1/\xi} + \left(\eta_\beta \beta\right)^{1/\xi}\right] \tag{C.26}$$

$$k_a + k_b = k. \tag{C.27}$$

2) Substitute n_a into (C.19) and n_b into (C.20). In addition, the multipliers μ_1 and μ_2 can be eliminated. The system is collapsed into a new one with 9 equations in 9 unknowns (l_a, l_b), (k, k_a, k_b), (α, β), (x_a, x_b)

$$\alpha^{1/\xi}(1-\mu)\left(Y_a/l_a\right) = a\left[1+\lambda\left(1-\eta_\alpha\right)\right]\xi\eta_\alpha^{(1-1/\xi)} \tag{C.28}$$

$$\beta^{1/\xi}(1-\gamma)\left(Y_b/l_b\right) = b\left[1+\lambda\left(1-\eta_\beta\right)\right]\xi\eta_\beta^{(1-1/\xi)} \tag{C.29}$$

$$\alpha^{\theta} a^{\theta-1}(1-\mu)\left(Y_a/l_a\right) = \beta^{\theta} b^{\theta-1}(1-\gamma)\left(Y_b/l_b\right) \tag{C.30}$$

$$\alpha^{\theta} a^{\theta-1}\mu\left(Y_a/k_a\right) = \beta^{\theta} b^{\theta-1}\gamma\left(Y_b/k_b\right) \tag{C.31}$$

$$1 = \lambda\mu\left(Y_a/k_a\right)\nu\left(\phi x_a^{\nu-1} x_b^{1-\nu}\right) \tag{C.32}$$

$$1 = \lambda\gamma\left(Y_b/k_b\right)(1-\nu)\left(\phi x_a^{\nu} x_b^{-\nu}\right) \tag{C.33}$$

$$\phi x_a^{\nu} x_b^{1-\nu} = \delta k \tag{C.34}$$

$$l_a + l_b = 1 - \left[\left(\eta_\alpha \alpha\right)^{1/\xi} + \left(\eta_\beta \beta\right)^{1/\xi}\right] \tag{C.35}$$

$$k_a + k_b = k. \tag{C.36}$$

3) We transform some variables to simplify the system. Let

$$S = k_b/k \tag{C.37}$$

$$Q = k_b/l_b \tag{C.38}$$

$$X_a = x_a / Y_a \tag{C.39}$$

$$X_b = x_b / Y_b. \tag{C.40}$$

From (C.30) and (C.31)

$$\frac{k_b}{k_a} = \left[\frac{\gamma(1-\mu)}{\mu(1-\gamma)}\right]\frac{l_b}{l_a}. \tag{C.41}$$

From (C.38) and (C.41)

$$\frac{k_a}{l_a} = \left[\frac{\mu(1-\gamma)}{\gamma(1-\mu)}\right] Q. \tag{C.42}$$

From (C.42)

$$\frac{Y_a}{l_a} = A\left[\frac{\mu(1-\gamma)}{\gamma(1-\mu)}\right]^{\mu} Q^{\mu}, \tag{C.43}$$

$$\frac{Y_a}{k_a} = A\left[\frac{\mu(1-\gamma)}{\gamma(1-\mu)}\right]^{\mu-1} Q^{\mu-1}. \tag{C.44}$$

From (C.38)

$$\frac{Y_b}{l_b} = BQ^{\gamma}, \tag{C.45}$$

$$\frac{Y_b}{k_b} = BQ^{\gamma-1}. \tag{C.46}$$

From (C.37) and (C.41), we have

$$\frac{S}{1-S} = \left[\frac{\gamma(1-\mu)}{\mu(1-\gamma)}\right]\frac{l_b}{l_a}$$

$$\frac{l_a}{\gamma(1-\mu)(1-S)} = \frac{l_b}{\mu(1-\gamma)S} = \frac{1-\left[(\eta_\alpha\alpha)^{1/\xi} + (\eta_\beta\beta)^{1/\xi}\right]}{\gamma(1-\mu)+(\mu-\gamma)S}$$

$$l_a = \frac{\gamma(1-\mu)(1-S)}{\gamma(1-\mu)+(\mu-\gamma)S}\left\{1-\left[(\eta_\alpha\alpha)^{1/\xi} + (\eta_\beta\beta)^{1/\xi}\right]\right\}, \tag{C.47}$$

$$l_b = \frac{\mu(1-\gamma)S}{\gamma(1-\mu)+(\mu-\gamma)S}\left\{1-\left[(\eta_\alpha\alpha)^{1/\xi} + (\eta_\beta\beta)^{1/\xi}\right]\right\}. \tag{C.48}$$

From (C.28) and (C.29)

$$a=\frac{\alpha^{1/\xi}\left(1-\mu\right)\left(Y_a/l_a\right)}{\left[1+\lambda\left(1-\eta_\alpha\right)\right]\xi\eta_\alpha^{(1-1/\xi)}}, \tag{C.49}$$

$$b=\frac{\beta^{1/\xi}\left(1-\gamma\right)\left(Y_b/l_b\right)}{\left[1+\lambda\left(1-\eta_\beta\right)\right]\xi\eta_\beta^{(1-1/\xi)}}. \tag{C.50}$$

Based on (C.43)-(C.46) and (C.49)-(C.50), equation (C.31) can be rewritten as

$$\frac{\alpha^\theta\mu\left(Y_a/k_a\right)}{\beta^\theta\gamma\left(Y_b/k_b\right)}=\frac{b^{\theta-1}}{a^{\theta-1}}$$

$$\frac{\alpha^\theta\mu}{\beta^\theta\gamma}\frac{\left(Y_a/k_a\right)}{\left(Y_b/k_b\right)}=\left[\frac{\left(1-\mu\right)\left[1+\lambda\left(1-\eta_\beta\right)\right]\eta_\beta^{(1-1/\xi)}}{\left(1-\gamma\right)\left[1+\lambda\left(1-\eta_\alpha\right)\right]\eta_\alpha^{(1-1/\xi)}}\frac{\alpha^{1/\xi}}{\beta^{1/\xi}}\frac{\left(Y_a/l_a\right)}{\left(Y_b/l_b\right)}\right]^{1-\theta}$$

$$\frac{\left(Y_a/k_a\right)}{\left(Y_b/k_b\right)}\frac{\left(Y_a/l_a\right)^{\theta-1}}{\left(Y_b/l_b\right)^{\theta-1}}\frac{\alpha^\theta}{\beta^\theta}=\frac{\gamma}{\mu}\left[\frac{\left(1-\mu\right)\left[1+\lambda\left(1-\eta_\beta\right)\right]\eta_\beta^{(1-1/\xi)}}{\left(1-\gamma\right)\left[1+\lambda\left(1-\eta_\alpha\right)\right]\eta_\alpha^{(1-1/\xi)}}\frac{\alpha^{1/\xi}}{\beta^{1/\xi}}\right]^{1-\theta}$$

$$Q^{\theta(\mu-\gamma)}\left(\frac{\alpha}{\beta}\right)^{\theta-(1-\theta)/\xi}=C_1, \tag{C.51}$$

where

$$C_1=\frac{B^\theta}{A^\theta}\left[\frac{\left[1+\lambda\left(1-\eta_\beta\right)\right]\eta_\beta^{(1-1/\xi)}}{\left[1+\lambda\left(1-\eta_\alpha\right)\right]\eta_\alpha^{(1-1/\xi)}}\right]^{1-\theta}\frac{\gamma^{\theta\mu}\left(1-\mu\right)^{\theta(\mu-1)}}{\mu^{\theta\mu}\left(1-\gamma\right)^{\theta(\mu-1)}}. \tag{C.52}$$

Equation (C.32) can be rewritten as

$$1=\frac{\lambda\mu\left(Y_a/k_a\right)\nu\left(\phi x_a^\nu x_b^{1-\nu}\right)}{x_a}$$

$$\frac{\left(Y_a/k_a\right)\nu\left(\phi x_a^\nu x_b^{1-\nu}\right)}{X_aY_a}=\frac{1}{\lambda\mu}$$

$$\frac{\nu\delta k}{X_ak_a}=\frac{1}{\lambda\mu}$$

$$X_a=\frac{\nu\delta\lambda\mu}{1-S}. \tag{C.53}$$

Equation (C.33) can be rewritten as

$$
\begin{aligned}
1 &= \frac{\lambda\gamma\left(Y_b/k_b\right)(1-\nu)\left(\phi x_a^{\nu}x_b^{1-\nu}\right)}{x_b}\\
\frac{\left(Y_b/k_b\right)(1-\nu)\left(\phi x_a^{\nu}x_b^{1-\nu}\right)}{X_bY_b} &= \frac{1}{\lambda\gamma}\\
\frac{(1-\nu)\,\delta k}{X_bk_b} &= \frac{1}{\lambda\gamma}\\
X_b &= \frac{(1-\nu)\,\delta\lambda\gamma}{S}. \qquad \text{(C.54)}
\end{aligned}
$$

4) We now solve for Q. Equation (C.34) can be rewritten as

$$
\phi\left(X_aY_a\right)^{\nu}\left(X_bY_b\right)^{1-\nu} = \delta k \qquad \text{(C.55)}
$$

$$
\phi\left[\frac{\nu\delta\lambda\mu}{1-S}\right]^{\nu}\left[\frac{(1-\nu)\,\delta\lambda\gamma}{S}\right]^{1-\nu}\left(\frac{Y_a}{k_a}k_a\right)^{\nu}\left(\frac{Y_b}{k_b}k_b\right)^{1-\nu} = \delta k
$$

$$
\lambda\phi A^{\nu}B^{1-\nu}\nu^{\nu}(1-\nu)^{1-\nu}\mu^{\mu\nu}\gamma^{1-\mu\nu}\left[\frac{1-\gamma}{1-\mu}\right]^{(\mu-1)\nu}Q^{\Lambda} = 1.
$$

Thus

$$
Q = C_2, \qquad \text{(C.56)}
$$

where

$$
C_2 = \left[\lambda\phi A^{\nu}B^{1-\nu}\nu^{\nu}(1-\nu)^{1-\nu}\mu^{\mu\nu}\gamma^{1-\mu\nu}\left[\frac{1-\gamma}{1-\mu}\right]^{(\mu-1)\nu}\right]^{-1/\Lambda}, \qquad \text{(C.57)}
$$

$$
\Lambda = \nu(\mu-1) + (1-\nu)(\gamma-1) \neq 0.
$$

5) We find the expression for β/α. From (C.51) and (C.56)

$$
\begin{aligned}
\left(\frac{\beta}{\alpha}\right)^{\theta-(1-\theta)/\xi} &= \frac{C_2^{\theta(\mu-\gamma)}}{C_1}\\
\frac{\beta}{\alpha} &= C_3, \qquad \text{(C.58)}
\end{aligned}
$$

where

$$C_3 = \left[\frac{C_2^{\theta(\mu-\gamma)}}{C_1}\right]^{1/[\theta-(1-\theta)/\xi]}, \ \theta(1+\xi) \neq 1. \tag{C.59}$$

6) We will find (S, X_a, X_b). Equations (C.28) and (C.29) are rewritten as

$$\frac{\alpha^{1/\xi}(1-\mu)}{[1+\lambda(1-\eta_\alpha)]\,\xi\eta_\alpha^{(1-1/\xi)}} = (1-X_a)\,l_a, \tag{C.60}$$

$$\frac{\beta^{1/\xi}(1-\gamma)}{[1+\lambda(1-\eta_\beta)]\,\xi\eta_\beta^{(1-1/\xi)}} = (1-X_b)\,l_b. \tag{C.61}$$

Based on (C.47)-(C.48) and (C.53)-(C.54), we divide (C.61) by (C.60)

$$\frac{[1+\lambda(1-\eta_\alpha)]\,\eta_\alpha^{(1-1/\xi)}(1-\gamma)}{[1+\lambda(1-\eta_\beta)]\,\eta_\beta^{(1-1/\xi)}(1-\mu)}\left(\frac{\beta}{\alpha}\right)^{1/\xi} = \frac{\left[1-\frac{(1-\nu)\delta\lambda\gamma}{S}\right]}{\left[1-\frac{\nu\delta\lambda\mu}{1-S}\right]}\frac{\mu(1-\gamma)S}{\gamma(1-\mu)(1-S)}$$

$$\frac{\gamma[1+\lambda(1-\eta_\alpha)]\,\eta_\alpha^{(1-1/\xi)}}{\mu[1+\lambda(1-\eta_\beta)]\,\eta_\beta^{(1-1/\xi)}}(C_3)^{1/\xi} = \frac{[S-(1-\nu)\delta\lambda\gamma]}{[1-S-\nu\delta\lambda\mu]}.$$

Let

$$C_4 = \frac{\gamma[1+\lambda(1-\eta_\alpha)]\,\eta_\alpha^{(1-1/\xi)}}{\mu[1+\lambda(1-\eta_\beta)]\,\eta_\beta^{(1-1/\xi)}}(C_3)^{1/\xi}, \tag{C.62}$$

and the expression for S follows

$$\frac{S-(1-\nu)\delta\lambda\gamma}{1-S-\nu\delta\lambda\mu} = C_4$$

$$S = \frac{C_4(1-\nu\delta\lambda\mu)+(1-\nu)\delta\lambda\gamma}{1+C_4}. \tag{C.63}$$

Plug S into (C.53) to find X_a and into (C.54) to find X_b.

7) At this point, all of the transformed variables are known, i.e. S, Q, X_a, X_b can be defined as explicit functions of the parameters. Other variables can now be derived. From (C.47) and (C.60), we have

$$\alpha^{1/\xi} = C_5\left\{1-\left[(\eta_\alpha\alpha)^{1/\xi}+(\eta_\beta\beta)^{1/\xi}\right]\right\},$$

where

$$C_5 = \frac{\left[1+\lambda\left(1-\eta_\alpha\right)\right]\xi\eta_\alpha^{(1-1/\xi)}\left(1-X_a\right)\gamma\left(1-S\right)}{\gamma\left(1-\mu\right)+\left(\mu-\gamma\right)S}; \tag{C.64}$$

and

$$\alpha^{1/\xi} = C_5 - C_5\left(\eta_\alpha\right)^{1/\xi}\alpha^{1/\xi} - C_5\left(\eta_\beta C_3\right)^{1/\xi}\alpha^{1/\xi}$$

$$\left[1+C_5\left(\eta_\alpha\right)^{1/\xi}+C_5\left(\eta_\beta C_3\right)^{1/\xi}\right]\alpha^{1/\xi} = C_5.$$

Thus, the steady-state quality indices are

$$\alpha = \left[\frac{C_5}{1+C_5\left(\eta_\alpha\right)^{1/\xi}+C_5\left(\eta_\beta C_3\right)^{1/\xi}}\right]^\xi, \tag{C.65}$$

$$\beta = C_3\left[\frac{C_5}{1+C_5\left(\eta_\alpha\right)^{1/\xi}+C_5\left(\eta_\beta C_3\right)^{1/\xi}}\right]^\xi. \tag{C.66}$$

Based on α and β, we can back out n_a from (C.23), n_b from (C.24), l_a from (C.47), and l_b from (C.48). Next, we can compute Y_a from (C.43), Y_b from (C.45), a from (C.49), and b from (C.50). Equations (C.39) and (C.40) generate x_a and x_b. Based on (C.55), we know k. Next are k_a and k_b from (C.37), μ_1 from (C.15) and μ_2 from (C.17).

8) The wage and interest rate are derived here. Normalized $p_a = 1$. The equilibrium wage is based on (C.43) and interest rate on (C.44)

$$w = \left(1-\mu\right)\frac{p_a Y_a}{l_a} = A\left(1-\mu\right)\left[\frac{\mu\left(1-\gamma\right)}{\gamma\left(1-\mu\right)}\right]^{\mu-1}Q^\mu, \tag{C.67}$$

$$r = \mu\frac{p_a Y_a}{k_a} = A\mu\left[\frac{\mu\left(1-\gamma\right)}{\gamma\left(1-\mu\right)}\right]^{\mu-1}Q^{\mu-1}. \tag{C.68}$$

9) There are several objects of special interest: b/a, Y_b/Y_a, β/α, p_{ba}, S_b. We know β/α from (C.58). From (C.43), (C.45), (C.49)-(C.50), and (C.58)

$$\frac{b}{a} = \frac{\left(1-\gamma\right)\left[1+\lambda\left(1-\eta_\alpha\right)\right]\eta_\alpha^{(1-1/\xi)}}{\left(1-\mu\right)\left[1+\lambda\left(1-\eta_\beta\right)\right]\eta_\beta^{(1-1/\xi)}}\left(\frac{\beta}{\alpha}\right)^{1/\xi}\frac{\left(Y_b/l_b\right)}{\left(Y_a/l_a\right)}$$

$$\frac{b}{a} = \left[\frac{\gamma^{\mu} (1-\gamma)^{1-\mu} [1+\lambda (1-\eta_{\alpha})] \eta_{\alpha}^{(1-1/\xi)}}{\mu^{\mu} (1-\mu)^{1-\mu} [1+\lambda (1-\eta_{\beta})] \eta_{\beta}^{(1-1/\xi)}} \right] \frac{B}{A} (C_3)^{1/\xi} C_2^{\gamma-\mu}. \tag{C.69}$$

From (C.37), (C.44) and (C.46)

$$\frac{Y_b}{Y_a} = \left[\frac{\mu (1-\gamma)}{\gamma (1-\mu)} \right]^{1-\mu} \frac{BQ^{\gamma-1} k_b}{AQ^{\mu-1} k_a}$$

$$\frac{Y_b}{Y_a} = \left[\frac{\mu (1-\gamma)}{\gamma (1-\mu)} \right]^{1-\mu} \frac{B}{A} C_2^{\gamma-\mu} \frac{[C_4 (1-\nu\delta\lambda\mu) + (1-\nu) \delta\lambda\gamma]}{[1 + C_4 \nu\delta\lambda\mu - (1-\nu) \delta\lambda\gamma]}. \tag{C.70}$$

From (C.58) and (C.69), the steady-state equilibrium relative price is

$$p_{ba} = \frac{\partial u (a, b; \alpha, \beta) / \partial b}{\partial u (a, b; \alpha, \beta) / \partial a} = \frac{c^{-\sigma} d\beta^{\theta} b^{\theta-1}}{c^{-\sigma} d\alpha^{\theta} a^{\theta-1}}$$

$$p_{ba} = \left(\frac{\beta}{\alpha} \right)^{\theta} \left(\frac{b}{a} \right)^{\theta-1} \tag{C.71}$$

$$p_{ba} = \left[\frac{\gamma^{\mu} (1-\gamma)^{1-\mu} [1+\lambda (1-\eta_{\alpha})] \eta_{\alpha}^{(1-1/\xi)}}{\mu^{\mu} (1-\mu)^{1-\mu} [1+\lambda (1-\eta_{\beta})] \eta_{\beta}^{(1-1/\xi)}} \frac{B}{A} \right]^{\theta-1} \frac{C_3^{\theta-(1-\theta)/\xi}}{C_2^{(\gamma-\mu)(1-\theta)}}. \tag{C.72}$$

Based on (C.69) and (C.72)

$$S_b = \frac{p_b b}{p_a a + p_b b} = \frac{p_{ba} (b/a)}{1 + p_{ba} (b/a)}$$

$$S_b = \frac{C_6}{1 + C_6}, \tag{C.73}$$

$$C_6 = \left[\frac{\gamma^{\mu} (1-\gamma)^{1-\mu} [1+\lambda (1-\eta_{\alpha})] \eta_{\alpha}^{(1-1/\xi)}}{\mu^{\mu} (1-\mu)^{1-\mu} [1+\lambda (1-\eta_{\beta})] \eta_{\beta}^{(1-1/\xi)}} \right]^{\theta} \left(\frac{B}{A} \right)^{\theta} \frac{C_3^{\theta(1+1/\xi)}}{C_2^{(\mu-\gamma)\theta}}. \tag{C.74}$$

These five objects are functions of C_1, C_2, and C_3, where (conditional on A)

$$\begin{aligned} C_1 &= const \times (B/A)^{\theta} \\ C_2 &= const \times (B/A)^{-(1-\nu)/\Lambda} \\ C_3 &= const \times (B/A)^{(1-\mu)\theta/\{[\theta-(1-\theta)/\xi]\Lambda\}}. \end{aligned}$$

After some manipulations, we observe the following qualitative effects

$$\begin{aligned} sign\left[\partial\left(b/a\right)/\partial\left(B/A\right)\right] &= sign\left[\left(1-\theta\xi\right)\left(\theta\xi+\theta-1\right)\Lambda\right] && \text{(C.75)} \\ sign\left[\partial\left(\beta/\alpha\right)/\partial\left(B/A\right)\right] &= sign\left[\theta\left(\theta\xi+\theta-1\right)\Lambda\right] && \text{(C.76)} \\ sign\left[\partial p_{ba}/\partial\left(B/A\right)\right] &= sign\left[\Lambda\right] && \text{(C.77)} \\ sign\left[\partial S_b/\partial\left(B/A\right)\right] &= sign\left[\theta\left(\theta\xi+\theta-1\right)\Lambda\right]. && \text{(C.78)} \end{aligned}$$

C.3 Log-linearization and Laws of Motion

To approximate the laws of motion of the endogenous objects, we first log-linearize the system of FOCs (C.5)-(C.14). A transformed variable is interpreted as percentage deviation of the original variable from the corresponding nonstochastic steady state value. In notation, $\widehat{x}_t = \log x_t - \log x$, where x_t is the original variable, $\widehat{x}_t$ is the transformed version, and x is the steady state value. For small deviations, $\widehat{x}_t \approx (x_t - x)/x$, and $f(x_t) = f(x) + f'(x)x\widehat{x}_t$. Evolution of the endogenous state variables $\{\alpha_{t-1}, \beta_{t-1}, k_t\}$ are specified in (4.6)-(4.8). The law of motion for the exogenous state variables $\{A_t, B_t\}$ is assumed to follow the VAR structure

$$\left[\begin{array}{c} A_t \\ B_t \end{array}\right] = \left[\begin{array}{c} \mu_a\left(1-\rho_a\right) \\ \mu_b\left(1-\rho_b\right) \end{array}\right] + \left[\begin{array}{cc} \rho_a & 0 \\ 0 & \rho_b \end{array}\right]\left[\begin{array}{c} A_{t-1} \\ B_{t-1} \end{array}\right] + \left[\begin{array}{c} \varepsilon_{at} \\ \varepsilon_{bt} \end{array}\right], \tag{C.79}$$

where $A_t \sim N(\mu_a, \sigma_a^2)$; $B_t \sim N(\mu_b, \sigma_b^2)$; and $\varepsilon_t = [\varepsilon_{at}\ \varepsilon_{bt}]' \sim N(\mathbf{0}, \Sigma)$ with

$$\mathbf{\Sigma} = \left[\begin{array}{cc} \left(1-\rho_a^2\right)\sigma_a^2 & 0 \\ 0 & \left(1-\rho_b^2\right)\sigma_b^2 \end{array}\right]. \tag{C.80}$$

We can substitute out the multipliers μ_{1t} and μ_{2t}. After some simple manipulations, the original FOC becomes

$$E_t\left\{\begin{array}{c} c_t^{-\sigma}d_t a_t^{\theta}\alpha_t^{\theta-1}\xi\left(n_{at}\right)^{\xi-1} - c_t^{-\sigma}d_t\alpha_t^{\theta}a_t^{\theta-1}\left(1-\mu\right)Y_{at}/l_{at} \\ +\lambda c_{t+1}^{-\sigma}d_{t+1}a_{t+1}^{\theta}\alpha_{t+1}^{\theta-1}\left(1-\eta_\alpha\right)\xi\left(n_{at}\right)^{\xi-1} \end{array}\right\} = 0 \tag{C.81}$$

$$E_t\left\{\begin{array}{c} c_t^{-\sigma}d_t b_t^{\theta}\beta_t^{\theta-1}\xi\left(n_{bt}\right)^{\xi-1} - c_t^{-\sigma}d_t\beta_t^{\theta}b_t^{\theta-1}\left(1-\gamma\right)Y_{bt}/l_{bt} \\ +\lambda c_{t+1}^{-\sigma}d_{t+1}b_{t+1}^{\theta}\beta_{t+1}^{\theta-1}\left(1-\eta_\beta\right)\xi\left(n_{bt}\right)^{\xi-1} \end{array}\right\} = 0 \tag{C.82}$$

$$E_t \left\{ \begin{array}{c} c_t^{-\sigma} d_t \alpha_t^{\theta} a_t^{\theta-1} (1-\mu) Y_{at}/l_{at} \\ -c_t^{-\sigma} d_t \beta_t^{\theta} b_t^{\theta-1} (1-\gamma) Y_{bt}/l_{bt} \end{array} \right\} = 0 \tag{C.83}$$

$$E_t \left\{ \begin{array}{c} c_t^{-\sigma} d_t \alpha_t^{\theta} a_t^{\theta-1} \mu Y_{at}/k_{at} \\ -c_t^{-\sigma} d_t \beta_t^{\theta} b_t^{\theta-1} \gamma Y_{bt}/k_{bt} \end{array} \right\} = 0 \tag{C.84}$$

$$E_t \left\{ \begin{array}{c} c_t^{-\sigma} d_t \alpha_t^{\theta} a_t^{\theta-1} \\ -\lambda c_{t+1}^{-\sigma} d_{t+1} \alpha_{t+1}^{\theta} a_{t+1}^{\theta-1} \mu \left(Y_{at+1}/k_{at+1}\right) \nu \left(\phi x_{at}^{\nu-1} x_{bt}^{1-\nu}\right) \end{array} \right\} = 0 \tag{C.85}$$

$$E_t \left\{ \begin{array}{c} c_t^{-\sigma} d_t \beta_t^{\theta} b_t^{\theta-1} \\ -\lambda c_{t+1}^{-\sigma} d_{t+1} \beta_{t+1}^{\theta} b_{t+1}^{\theta-1} \gamma \left(Y_{bt+1}/k_{bt+1}\right) (1-\nu) \left(\phi x_{at}^{\nu} x_{bt}^{-\nu}\right) \end{array} \right\} = 0, \tag{C.86}$$

and (4.4)-(4.10). In every period t, given the state $\{\alpha_{t-1}, \beta_{t-1}, k_t, A_t, B_t\}$, the set of endogenous variables can be collapsed into a minimal set, based on which all the remaining equilibrium objects can be backed out. We choose the minimal set $\{\alpha_t, \beta_t, k_{t+1}, l_{bt}, k_{bt}, x_{bt}\}$. Recall that there are more choice variables. From (4.4)-(4.8), the remaining decisions can be defined as follows: $n_{at} = [\alpha_t - (1-\eta_\alpha)\alpha_{t-1}]^{1/\xi}$; $n_{bt} = [\beta_t - (1-\eta_\beta)\beta_{t-1}]^{1/\xi}$; $l_{at} = 1 - l_{bt} - n_{at} - n_{bt}$; $k_{at} = k_t - k_{bt}$; and $x_{at} = [(k_{t+1} - (1-\delta)k_t)/(\phi x_{bt}^{1-\nu})]^{1/\nu}$. Based on (C.81)-(C.86), we construct a log-linearized system of expectation difference equations

$$E_t \left\{\zeta_0 z_{t+1} + \zeta_1 z_t + \zeta_2 z_{t-1} + \kappa_0 s_{t+1} + \kappa_1 s_t\right\} \tag{C.87}$$

where $z_t = [\widehat{\alpha}_t \; \widehat{\beta}_t \; \widehat{k}_{t+1} \; \widehat{l}_{bt} \; \widehat{k}_{bt} \; \widehat{x}_{bt}]'$; $s_t = [\widehat{A}_t \; \widehat{\beta}_t]'$; $\{\zeta_0, \zeta_1, \zeta_2\}$ are 6×6 matrices; and $\{\kappa_0, \kappa_1\}$ are 6×2 matrices. In fact, these matrices are the first-order derivatives of the system (C.81)-(C.86) with respect to the corresponding variables in (C.87), and then rescaled by steady state values. Though we can derive these matrices analytically, it is much easier to compute them.

Let $\Gamma(6\times6)$ and $\Psi(6\times2)$ respectively be the feedback and feedforward matrices. The linear laws of motion of the endogenous objects have the form

$$z_t = \Gamma z_{t-1} + \Psi s_t \tag{C.88}$$

$$s_t = \rho s_{t-1} + \varepsilon_t, \tag{C.89}$$

where matrix ρ is derived from log-linearization of (C.79); Γ and Ψ are unknown and solved by the method of undetermined coefficients (Christiano 2002). More specifically, Γ and Ψ are respectively solutions of the matrix equations

$$\zeta_0\Gamma^2 + \zeta_1\Gamma + \zeta_2 = 0 \tag{C.90}$$

$$\zeta_0\Gamma\Psi + \zeta_0\Psi\rho + \zeta_1\Psi + \kappa_0\rho + \kappa_1. \tag{C.91}$$

In Section 4.3, given the specified parameters, these two matrices, which correspond to different values of θ, given $\xi = 2/3$, are

$$\Gamma_{\theta=-3} = \begin{bmatrix} 0.75 & 0.06 & 0.02 & 0 & 0 & 0 \\ 0.06 & 0.71 & 0.03 & 0 & 0 & 0 \\ 0.10 & 0.12 & 0.57 & 0 & 0 & 0 \\ 0.39 & -0.08 & -0.08 & 0 & 0 & 0 \\ 0.27 & -0.26 & 1.00 & 0 & 0 & 0 \\ 1.95 & 2.40 & -7.54 & 0 & 0 & 0 \end{bmatrix}, \quad \Psi_{\theta=-3} = \begin{bmatrix} -0.06 & 0.03 \\ 0.03 & -0.07 \\ 0.34 & 0.44 \\ 0.38 & -0.25 \\ 0.37 & -0.35 \\ 6.42 & 9.19 \end{bmatrix};$$

$$\Gamma_{\theta=0.3} = \begin{bmatrix} 0.81 & -0.00 & 0.03 & 0 & 0 & 0 \\ -0.01 & 0.78 & 0.03 & 0 & 0 & 0 \\ 0.11 & 0.11 & 0.57 & 0 & 0 & 0 \\ -0.02 & 0.31 & -0.10 & 0 & 0 & 0 \\ -0.17 & 0.18 & 0.98 & 0 & 0 & 0 \\ 2.15 & 2.23 & -7.53 & 0 & 0 & 0 \end{bmatrix}, \quad \Psi_{\theta=0.3} = \begin{bmatrix} 0.02 & -0.05 \\ -0.05 & 0.01 \\ 0.36 & 0.43 \\ -0.15 & 0.27 \\ -0.23 & 0.25 \\ 6.72 & 9.07 \end{bmatrix};$$

$$\Gamma_{\theta=.59} = \begin{bmatrix} 0.97 & -0.16 & 0.04 & 0 & 0 & 0 \\ -0.02 & 0.79 & 0.03 & 0 & 0 & 0 \\ 0.03 & 0.20 & 0.57 & 0 & 0 & 0 \\ -0.08 & 0.34 & -0.03 & 0 & 0 & 0 \\ -0.13 & 0.09 & 1.05 & 0 & 0 & 0 \\ 0.54 & 4.04 & -7.71 & 0 & 0 & 0 \end{bmatrix}, \quad \Psi_{\theta=.59} = \begin{bmatrix} 0.25 & -0.24 \\ -0.04 & 0.01 \\ 0.31 & 0.54 \\ -0.12 & 0.10 \\ -0.21 & 0.08 \\ 5.65 & 11.20 \end{bmatrix}.$$

C.4 Quality-augmented Capital

With the quality-augmented capital hypothesis, the set of state variables has one more element $\{\alpha_{t-1}, \beta_{t-1}, \overline{q}_t, k_t, A_t, B_t\}$ where $\overline{q}_t$ evolves according to (4.22); and the production functions become (4.23) and (4.24) where average quality augments capital services. The dynamic programming problem is again defined as in (C.1). We use the shorthands $\{c_t, d_t\}$ in (C.2) and (C.3), and define several new objects

$$
\begin{aligned}
DA_{t+1}\left(n_{at}\right) &= \frac{1}{\theta}\frac{\partial\left(a_{t+1}^{\theta}\alpha_{t+1}^{\theta}\right)}{\partial n_{at}} \\
&= \alpha_{t+1}^{\theta}a_{t+1}^{\theta-1}\frac{\mu Y_{at+1}}{\overline{q}_{t+1}}\left(S_k\nu\alpha_t^{\nu-1}\beta_t^{1-\nu}\right)\left(\xi n_{at}^{\xi-1}\right) \\
&\quad +a_{t+1}^{\theta}\alpha_{t+1}^{\theta-1}\left(1-\eta_\alpha\right)\left(\xi n_{at}^{\xi-1}\right),
\end{aligned}
\tag{C.92}
$$

$$
\begin{aligned}
DB_{t+1}\left(n_{bt}\right) &= \frac{1}{\theta}\frac{\partial\left(b_{t+1}^{\theta}\beta_{t+1}^{\theta}\right)}{\partial n_{bt}} \\
&= \beta_{t+1}^{\theta}b_{t+1}^{\theta-1}\frac{\gamma Y_{bt+1}}{\overline{q}_{t+1}}\left(S_k\left(1-\nu\right)\alpha_t^{\nu}\beta_t^{-\nu}\right)\left(\xi n_{bt}^{\xi-1}\right) \\
&\quad +b_{t+1}^{\theta}\beta_{t+1}^{\theta-1}\left(1-\eta_\beta\right)\left(\xi n_{bt}^{\xi-1}\right).
\end{aligned}
\tag{C.93}
$$

These expressions are related to the future marginal gains of current quality investment via capital augmentation. With these notations, the full system of FOC is

$$c_t^{-\sigma}d_t\alpha_t^{\theta}a_t^{\theta-1}\left(1-\mu\right)Y_{at}/l_{at}=\mu_{1t} \tag{C.94}$$

$$c_t^{-\sigma}d_t\beta_t^{\theta}b_t^{\theta-1}\left(1-\gamma\right)Y_{bt}/l_{bt}=\mu_{1t} \tag{C.95}$$

$$c_t^{-\sigma}d_t\alpha_t^{\theta}a_t^{\theta-1}\mu Y_{at}/k_{at}=\mu_{2t} \tag{C.96}$$

$$c_t^{-\sigma}d_t\beta_t^{\theta}b_t^{\theta-1}\gamma Y_{bt}/k_{bt}=\mu_{2t} \tag{C.97}$$

$$c_t^{-\sigma}d_ta_t^{\theta}\alpha_t^{\theta-1}\xi\left(n_{at}\right)^{\xi-1}+\lambda E_t\left[c_{t+1}^{-\sigma}d_{t+1}DA_{t+1}\left(n_{at}\right)\right]=\mu_{1t} \tag{C.98}$$

$$c_t^{-\sigma}d_tb_t^{\theta}\beta_t^{\theta-1}\xi\left(n_{bt}\right)^{\xi-1}+\lambda E_t\left[c_{t+1}^{-\sigma}d_{t+1}DB_{t+1}\left(n_{bt}\right)\right]=\mu_{1t} \tag{C.99}$$

$$
\begin{gathered}
c_t^{-\sigma}d_t\alpha_t^{\theta}a_t^{\theta-1}= \\
+\lambda E_t\left[c_{t+1}^{-\sigma}d_{t+1}\alpha_{t+1}^{\theta}a_{t+1}^{\theta-1}\mu\left(Y_{at+1}/k_{at+1}\right)\nu\left(\phi x_{at}^{\nu-1}x_{bt}^{1-\nu}\right)\right]
\end{gathered}
\tag{C.100}
$$

$$
\begin{gathered}
c_t^{-\sigma}d_t\beta_t^{\theta}b_t^{\theta-1}= \\
+\lambda E_t\left[c_{t+1}^{-\sigma}d_{t+1}\beta_{t+1}^{\theta}b_{t+1}^{\theta-1}\gamma\left(Y_{bt+1}/k_{bt+1}\right)\left(1-\nu\right)\left(\phi x_{at}^{\nu}x_{bt}^{-\nu}\right)\right]
\end{gathered}
\tag{C.101}
$$

$$l_{at}+l_{bt}+n_{at}+n_{bt}=1;\ \mu_{1t}>0 \tag{C.102}$$

$$k_{at} + k_{bt} = k_t;\ \mu_{2t} > 0. \tag{C.103}$$

The steady state of interest is $\{l_a, l_b, n_a, n_b, k, k_a, k_b, \alpha, \beta, \overline{q}, x_a, x_b\}$. To solve for these objects, we rely on (C.22)-(C.24) and the following 9 conditions

$$\alpha^{1/\xi}(1-\mu)\,Y_a/l_a = \{a\,[1+\lambda\,(1-\eta_\alpha)] + Y_a\lambda\mu S_k\nu\}\,\xi\eta_\alpha^{(1-1/\xi)} \tag{C.104}$$

$$\beta^{1/\xi}(1-\gamma)\,Y_b/l_b = \{b\,[1+\lambda\,(1-\eta_\beta)] + Y_b\lambda\gamma S_k\,(1-\nu)\}\,\xi\eta_\beta^{(1-1/\xi)} \tag{C.105}$$

$$\alpha^\theta a^{\theta-1}(1-\mu)\,(Y_a/l_a) = \beta^\theta b^{\theta-1}(1-\gamma)\,(Y_b/l_b) \tag{C.106}$$

$$\alpha^\theta a^{\theta-1}\mu\,(Y_a/k_a) = \beta^\theta b^{\theta-1}\gamma\,(Y_b/k_b) \tag{C.107}$$

$$1 = \lambda\mu\,(Y_a/k_a)\,\nu\left(\phi x_a^{\nu-1}x_b^{1-\nu}\right) \tag{C.108}$$

$$1 = \lambda\gamma\,(Y_b/k_b)\,(1-\nu)\left(\phi x_a^{\nu}x_b^{-\nu}\right) \tag{C.109}$$

$$\phi x_a^\nu x_b^{1-\nu} = \delta\,(k_a+k_b) \tag{C.110}$$

$$l_a + l_b = 1 - \left[(\eta_\alpha\alpha)^{1/\xi} + (\eta_\beta\beta)^{1/\xi}\right] \tag{C.111}$$

$$\overline{q} = \alpha^\nu\beta^{1-\nu}. \tag{C.112}$$

To approximate dynamic behavior around the steady state, we apply log-linearization and the method of undetermined coefficient once again. Let $z_t = [\widehat{\alpha}_t\ \widehat{\beta}_t\ \widehat{\overline{q}}_t\ \widehat{k}_{t+1}\ \widehat{l}_{bt}\ \widehat{k}_{bt}\ \widehat{x}_{bt}]'$ and $s_t = [\widehat{A}_t\ \widehat{\beta}_t]'$. The original system becomes

$$E_t\left\{\begin{array}{c} c_t^{-\sigma}d_t a_t^\theta\alpha_t^{\theta-1}\xi\,(n_{at})^{\xi-1} + \lambda c_{t+1}^{-\sigma}d_{t+1}DA_{t+1}\,(n_{at}) \\ -c_t^{-\sigma}d_t\alpha_t^\theta a_t^{\theta-1}(1-\mu)\,Y_{at}/l_{at}\end{array}\right\} = 0 \tag{C.113}$$

$$E_t\left\{\begin{array}{c} c_t^{-\sigma}d_t b_t^\theta\beta_t^{\theta-1}\xi\,(n_{bt})^{\xi-1} + \lambda c_{t+1}^{-\sigma}d_{t+1}DB_{t+1}\,(n_{bt}) \\ -c_t^{-\sigma}d_t\beta_t^\theta b_t^{\theta-1}(1-\gamma)\,Y_{bt}/l_{bt}\end{array}\right\} = 0 \tag{C.114}$$

$$E_t\left\{\begin{array}{c} c_t^{-\sigma}d_t\alpha_t^\theta a_t^{\theta-1}(1-\mu)\,Y_{at}/l_{at} \\ -c_t^{-\sigma}d_t\beta_t^\theta b_t^{\theta-1}(1-\gamma)\,Y_{bt}/l_{bt}\end{array}\right\} = 0 \tag{C.115}$$

$$E_t\left\{\begin{array}{c} c_t^{-\sigma}d_t\alpha_t^\theta a_t^{\theta-1}\mu Y_{at}/k_{at} \\ -c_t^{-\sigma}d_t\beta_t^\theta b_t^{\theta-1}\gamma Y_{bt}/k_{bt}\end{array}\right\} = 0 \tag{C.116}$$

$$E_t\left\{\begin{array}{c} c_t^{-\sigma}d_t\alpha_t^\theta a_t^{\theta-1} \\ -\lambda c_{t+1}^{-\sigma}d_{t+1}\alpha_{t+1}^\theta a_{t+1}^{\theta-1}\mu\,(Y_{at+1}/k_{at+1})\,\nu\left(\phi x_{at}^{\nu-1}x_{bt}^{1-\nu}\right)\end{array}\right\} = 0 \tag{C.117}$$

$$E_t\left\{\begin{array}{c} c_t^{-\sigma}d_t\beta_t^\theta b_t^{\theta-1} \\ -\lambda c_{t+1}^{-\sigma}d_{t+1}\beta_{t+1}^\theta b_{t+1}^{\theta-1}\gamma\,(Y_{bt+1}/k_{bt+1})\,(1-\nu)\left(\phi x_{at}^{\nu}x_{bt}^{-\nu}\right)\end{array}\right\} = 0 \tag{C.118}$$

$$E_t\left\{\overline{q}_{t+1} - (1-S_k)\,\overline{q}_t - S_k\alpha_t^\nu\beta_t^{1-\nu}\right\} = 0. \tag{C.119}$$

C.5 Technical Details of the US Application

Our main interest is the time path of US services-goods relative quality $\{Q\}_{t=1970}^{2006}$ which is driven by technology changes. To infer this unobservable time series, we need to find $\{\theta, \eta_\alpha, \eta_\beta, \xi\}$, $\{A_t, B_t\}_{t=1970}^{2006}$, and the initial state which make the model time paths $\{Y_{at}, Y_{bt}, p_t, S_{bt}\}_{t=1970}^{2006}$ have some moments close to those of data. The representative agent is assumed to perfectly observe and foresee the evolutions of A_t and B_t, i.e. there are no uncertainties.

Let the model time length be $T + 1$, which includes period 0 and the terminal period T. In T, there are no intertemporal benefits and hence no dynamic decisions. Thus we have the time line $[0, 1, .., S, S + 1, .., T]$, where 1 and S respectively correspond the years 1970 and 2006, i.e. $S = 36$. Relabeling the time line, we need to find $\{A_t, B_t\}_{t=1}^{T}$. By normalization, in the year 2000, $A_{S-5} = 1$ and $B_{S-5} = 0.98$. The entire productivity time series are constructed based on the geometric mean growth rates of $\{A_t, B_t\}$ in the sample period 1970-2006. Next, $\{\alpha_0, \beta_0, k_0\}$ are assumed to be at the steady state values if the economy has productivity levels $\{A_1, B_1\}$ forever.

Given $\{A_t, B_t\}_{t=1}^{T}$ and $\{\alpha_0, \beta_0, k_0\}$, we need to find the four critical parameters. Our algorithm has two levels, the higher for finding these unknown parameters and the lower for having the corresponding equilibrium. At the higher level, we start with some guess on $\{\theta, \eta_\alpha, \eta_\beta, \xi\}$; construct the differences between data and model moments; update the guess for the next round, and continue until the guess converges. At the lower level, given some $\{\theta, \eta_\alpha, \eta_\beta, \xi\}$, we need to solve a large system of equations, which characterizes the corresponding equilibrium for $[1, .., T]$. The end point T is chosen not very far from S so that productivity projections are reliable; and at the same time, far enough so that the solutions in $[1, .., S]$ are not significantly influenced by terminal decisions. In fact, there is a trade-off between these two objectives.

Here are some details about the lower level. First, $\{\theta,\ \eta_\alpha,\ \eta_\beta,\ \xi\}$, $\{\alpha_0,\ \beta_0,\ k_0\}$, $\{A_t, B_t\}_{t=1}^T$ are given numbers. Second, all the dynamic choices in period T are zero. Third, the state observed at the beginning of any period t consists of $\{\alpha_{t-1}, \beta_{t-1}, k_t, A_t, B_t\}$, out of which the first three are endogenous. Fourth, as $\nu \in (0,1)$, $x_{at} > 0$ and $x_{bt} > 0$; and for $\xi \in (0,1)$, $n_{at} > 0$ and $n_{bt} > 0$. We choose the minimal set of equilibrium objects in each period to be $\{l_{bt}, k_{bt}, n_{at}, n_{bt}, x_{at}, x_{bt}\}$, out of which the last four govern the evolution of the state. Fifth, we have a system of $6T$ nonlinear equations derived from (C.5) to (C.14). Specifically, the system of necessary conditions for $t \in [1, T-1]$ is

$$c_t^{-\sigma} d_t \beta_t^\theta b_t^{\theta-1} (1-\gamma) B_t k_{bt}^\gamma l_{bt}^{-\gamma} - c_t^{-\sigma} d_t \alpha_t^\theta a_t^{\theta-1} (1-\mu) A_t k_{at}^\mu l_{at}^{-\mu} = 0 \tag{C.120}$$

$$c_t^{-\sigma} d_t \beta_t^\theta b_t^{\theta-1} \gamma B_t k_{bt}^{\gamma-1} l_{bt}^{1-\gamma} - c_t^{-\sigma} d_t \alpha_t^\theta a_t^{\theta-1} \mu A_t k_{at}^{\mu-1} l_{at}^{1-\mu} = 0 \tag{C.121}$$

$$\begin{gathered} c_t^{-\sigma} d_t a_t^\theta \alpha_t^{\theta-1} \xi (n_{at})^{\xi-1} - c_t^{-\sigma} d_t \alpha_t^\theta a_t^{\theta-1} (1-\mu) A_t k_{at}^\mu l_{at}^{-\mu} \\ + \lambda c_{t+1}^{-\sigma} d_{t+1} a_{t+1}^\theta \alpha_{t+1}^{\theta-1} (1-\eta_\alpha) \xi (n_{at})^{\xi-1} = 0 \end{gathered} \tag{C.122}$$

$$\begin{gathered} c_t^{-\sigma} d_t b_t^\theta \beta_t^{\theta-1} \xi (n_{bt})^{\xi-1} - c_t^{-\sigma} d_t \beta_t^\theta b_t^{\theta-1} (1-\gamma) B_t k_{bt}^\gamma l_{bt}^{-\gamma} \\ + \lambda c_{t+1}^{-\sigma} d_{t+1} b_{t+1}^\theta \beta_{t+1}^{\theta-1} (1-\eta_\beta) \xi (n_{bt})^{\xi-1} = 0 \end{gathered} \tag{C.123}$$

$$\begin{gathered} \lambda c_{t+1}^{-\sigma} d_{t+1} \alpha_{t+1}^\theta a_{t+1}^{\theta-1} \mu A_{t+1} k_{at+1}^{\mu-1} l_{at+1}^{1-\mu} \nu \phi x_{at}^{\nu-1} x_{bt}^{1-\nu} \\ - c_t^{-\sigma} d_t \alpha_t^\theta a_t^{\theta-1} = 0 \end{gathered} \tag{C.124}$$

$$\begin{gathered} \lambda c_{t+1}^{-\sigma} d_{t+1} \beta_{t+1}^\theta b_{t+1}^{\theta-1} \gamma B_{t+1} k_{bt+1}^{\gamma-1} l_{bt+1}^{1-\gamma} (1-\nu) \phi x_{at}^\nu x_{bt}^{-\nu} \\ - c_t^{-\sigma} d_t \beta_t^\theta b_t^{\theta-1} = 0. \end{gathered} \tag{C.125}$$

For the terminal period T, there are some changes as follows: two equations (C.120) and (C.121) still hold; (C.122) and (C.123) respectively become

$$c_t^{-\sigma} d_t a_t^\theta \alpha_t^{\theta-1} \xi (n_{at})^{\xi-1} - c_t^{-\sigma} d_t \alpha_t^\theta a_t^{\theta-1} (1-\mu) A_t k_{at}^\mu l_{at}^{-\mu} = 0 \tag{C.126}$$

$$c_t^{-\sigma} d_t b_t^\theta \beta_t^{\theta-1} \xi (n_{bt})^{\xi-1} - c_t^{-\sigma} d_t \beta_t^\theta b_t^{\theta-1} (1-\gamma) B_t k_{bt}^\gamma l_{bt}^{-\gamma} = 0; \tag{C.127}$$

and (C.124)-(C.125) are changed to

$$x_{at} = 0 \tag{C.128}$$

$$x_{bt} = 0. \tag{C.129}$$

Thus the system effectively has $6T - 2$ nonlinear equations in $6T - 2$ unknowns.

At the upper level, the objective function for moment matching is

$$F(\tau) = (m(\tau) - m)'(m(\tau) - m), \tag{C.130}$$

where $\tau = \{\theta, \eta_\alpha, \eta_\beta, \xi\}$; and $\{m, m(\tau)\}$ are vectors of data and model moments, respectively. Based on the limited data set, we only have two data moments: one is the angle between the linear trends of B_t/A_t and S_{bt}, and one is the slope of S_{bt} trend. It is noted that, as predicted by Proposition 4.3.3, the relative price does not bear information of these parameters, i.e. simulated p_{bat} does not respond to changes in τ. In this model, we are more interested in $\{\theta, \xi\}$ than $\{\eta_\alpha, \eta_\beta\}$. To overcome the under-identification problem, we need to fix $\{\eta_\alpha, \eta_\beta\}$. We first impose that $\eta_\alpha = \eta_\beta$ and do experiments on the discrete parameter space. Based on the objective function, we see that $\{\eta_\alpha, \eta_\beta\}$ should be small and cannot be zero. Next, by perturbing $\{\eta_\alpha, \eta_\beta\}$, we find $\eta_\beta > \eta_\alpha$ should hold. Finally, $\eta_\alpha = 0.0045$ and $\eta_\beta = 0.0055$. Conditional on $\{\eta_\alpha, \eta_\beta\}$, we employ Newton's method to search for $\{\theta, \xi\}$.

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